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THE FIRST TWENTY YEARS OF THE TEACHING COMMITTEE.

BY A. W. SIDDONS

WHEN the A.I.G.T. was founded in 1871 its committee was really a teaching committee. By 1878 sub-committees had been appointed to consider different branches of Geometry; in 1884 sub-committees were appointed to consider Arithmetic and Mechanics. But it was in 1902 that the Mathematical Association first appointed a committee described as a "Teaching Committee". What was it that led up to the appointment of this teaching committee?

At the Glasgow meeting of the British Association in September 1901, Professor John Perry read a characteristic paper attacking the methods of teaching mathematics that were at that time in general use. The reasons he gave for reform were excellent, but the syllabus he put forward would have been quite impossible in the hands of the vast majority of teachers of that day. To Professor Perry we owe a great debt for the publicity he gave to the question, even if his syllabus was impossible. As a result of his paper the British Association appointed a committee of which Professor A. R. Forsyth was chairman and Professor Perry secretary, "to report upon improvements that might be made in the teaching of mathematics."

In August 1901 Charles Godfrey and I spent a long holiday together; he then told me that Forsyth had spoken of the probable appointment of the B. A. Committee and had suggested that he should send a letter to that Committee about possible reforms. We drew up such a letter which obtained considerable publicity and was known as "The letter of 23 Schoolmasters": it was printed in the *Mathematical Gazette*, Vol. 2, p. 143, and also, I think, in *Nature*. Among the signatories of that letter were Rawdon Levett, who first suggested the formation of the A.I.G.T. and was its secretary for its first 15 years, and E. M. Langley who became a member of the A.I.G.T. in 1882. But the majority of its signatories were young men under 30.

The letter suggested moderate reforms that would be possible at once in the teaching of Geometry, Algebra and Arithmetic and suggested the early introduction of numerical trigonometry.

In January 1902, Alfred Lodge, who was a member of the B. A. Committee, read a paper at the Annual Meeting of the M. A. on the reform of mathematical teaching which was followed by a long discussion. That paper and the letter

of the 23 schoolmasters were the main causes of the appointment of the first Teaching Committee.

The Committee, which was appointed with powers to coopt, included a good many older members of the M. A. and a fair sprinkling of younger men: by the time its first report was issued, after the cooption of several younger men, it consisted of 29 members, including 23 schoolmasters—7 of these and one Cambridge don were under 30 years of age. Of the signatories of the letter of 23 schoolmasters, Godfrey, Holmes, Langley, Tuckey and Siddons were members of the original committee. Tuckey, I believe, has been a member of the Committee from that day to this; he and I are the only two survivors of the original Committee who still take any part in the activities of the Association.

The first meeting of the Committee and all subsequent meetings up to January 1911 were held at King's College. After practically every other schoolmaster had refused to undertake the secretaryship, I was appointed secretary, a post I held for over 10 years and after which I became chairman for several years. My appointment gave great pleasure to my old master, Rawdon Levett, and caused him to hand over to me many of the papers etc. referring to the early days of the A.I.G.T.—most of these are now in the M.A. Library.

The Committee set itself two aims:

(i) To suggest reforms in mathematical teaching,

(ii) To persuade examining bodies to revise their syllabuses and papers so that the reforms could be made.

Naturally the letter of the 23 schoolmasters was most helpful. It was at once decided that the wise course was to suggest moderate reforms, in the hope that we could get many mathematical teachers to approve our proposals and that with a large body of support we should be able to bring much pressure to bear on examining bodies.

The Committee acted very quickly: the first four meetings were held within five weeks. We first attacked the question of Geometry and our first report was published in the *Mathematical Gazette* for May 1902 (p. 167)—a remarkable performance considering that the Committee had only been appointed on January 18th of that year. Our second report was on Algebra and Arithmetic and appeared in the *Mathematical Gazette* for July 1902.

In December 1902, at the instigation of Professor Forsyth, the Cambridge Senate appointed a Syndicate (i.e. a University Committee) to consider the mathematics in the pass examinations of the University. Barnard and I, from the Teaching Committee, were members of the Syndicate. The first thing considered by the Syndicate was the Geometry of the Previous Examination; to my surprise Professor Forsyth said that he, bolder than the Teaching Committee, would have nothing to do with any report that recommended the retention of Euclid's order of propositions. In the summer of 1903 the Syndicate's first report was adopted by the Senate and Euclid's order no longer dominated the teaching of geometry in England.

It seems rather strange that Euclid's order should have had the influence that it did. The first recommendation of the M. A. Geometry report, almost echoing the first recommendation of the A.I.G.T., read as follows: " A first introduction to Geometry should not be formal but experimental, with use of instruments and numerical measurements and calculations." Yet the majority of pupils started geometry by learning a string of definitions, postulates and axioms, then going on to learn propositions; in general no experimental work was done, no drawing. I started to learn geometry in that way in 1889, though King Edward's School, Birmingham was one of the schools in which better methods might have been expected to prevail. I remember that

Mr. Bushell in his presidential address said that in his last term at Charterhouse in 1903 he saw a protractor for the first time.

No doubt boys taking at that time examinations such as Oxford Responsions or the Cambridge Previous or the Local examinations were compelled to learn Euclid; but that is no reason why they should have been plunged straight into Euclid without some preliminary work. Were the Preparatory Schools to blame, or the Public Schools? Certainly the Public Schools gave no encouragement to the Preparatory Schools to start sensibly—entrance examinations to Public Schools were on Euclid pure and simple.

Much might have been done, yet little was, until Cambridge and other examining bodies freed themselves from Euclid's order.

Besides the relief from the bondage of Euclid's order, the report of the Syndicate, as well as that of the Teaching Committee, laid down that in the ordinary school course proportion should be treated for commensurable magnitudes only.

In this connection it is interesting that Professor M. J. M. Hill wrote pointing out that in the Teaching Committee's report it was assumed *without any justification* that

$$\frac{\text{rect. } BC\,.\,AG}{\text{rect. } EF\,.\,DH} = \frac{BC}{EF}\cdot\frac{AG}{DH}.$$

Subsequently Professor Hill resigned from the Committee on the grounds that he could not sign a report which recommended the exclusion of incommensurables from an ordinary school course.

As a constrast to this attitude, Professor Forsyth, at the Annual General Meeting of 1903, said that he regarded the teaching of the theory of incommensurables as a most advanced University subject, and that the idea of attempting to teach the theory to the average schoolboy seemed to him almost an impossibility.

Practically all examining bodies followed the report of the Cambridge Syndicate: they gave freedom from Euclid's order and adopted most of the recommendations of the Teaching Committee about Algebra and Arithmetic. In particular I must mention the great help given to reform by the syllabuses and papers of the Civil Service Commission.

I feel that the rather unexpected success in gaining freedom from the dominance of Euclid owed much to the spadework done by the A.I.G.T. in the 1870's, though strange to say no one seemed to have considered the old A.I.G.T. reports when the Teaching Committee was drawing up its Geometry report. I had never seen the A.I.G.T. reports at that time, yet the recommendations of 1902 were very much the same as those of the 1870's.

Though it may be said that the break-through in the matter of Geometry had been achieved, much mopping up remained to be done. Very many teachers and authors failed to grasp the aims of the reformers: the spirit of Euclid was not dead: on the one hand much time was still devoted to the learning of propositions, and on the other hand much time was wasted over accurate drawing without much aim.

Let me skip on a few years and mention the famous Board of Education Circular No. 711 of 1909. This circular set out the aims of the reformers and gave much excellent advice. It divided the teaching of Geometry into:

Stage A. Introductory work on the fundamental concepts, the acquisition of a geometrical vocabulary without formal definitions.

Stage B. The discovery, by experiment and intuition, of the fundamental facts of geometry including facts about angles, parallels and congruent triangles, without any attempt at teaching proofs of these, elementary ideas of logical argument.

Stage C. The deductive development of logical geometry.
This authoritative statement did much to consolidate the work begun by the Teaching Committee's first report on geometry.

In the years 1904 to 1910 the Teaching Committee produced reports on Mechanics, mathematics in Preparatory Schools, advanced school mathematics; and a notable joint report with the Science Masters Association on correlation of Mathematics and Science.

So far the Teaching Committee might be described as being chiefly concerned with the mathematics of Public Schools; but in January 1913 a change was made: separate committees were formed for Public Schools, for other Secondary Schools and for Girls' Schools, and a General Committee; about 1923 the committees for Public Schools and for other Secondary Schools were merged into one committee for Boys' Schools. Several reports were produced by these committees and passed for publication by the General Committee in the next few years.

Of all the reports published before 1918 it might be said that they consisted of short general statements of the aims of teaching, followed by brief statements of what should be included and what should be cut out of our teaching and examination syllabuses. I feel that this was the best plan while our first aim was to get examination papers altered. The range of practically all the pre-1918 reports might be described as that of the work for ordinary and additional mathematics in the old School certificate examination. Mathematical specialists were hardly considered. From 1918 onwards our reports have been of a different character: they have become admirable essays on the teaching of different subjects and, in general, addressed to the teacher rather than to the examiner. Their range has included the work of the mathematical specialist at school.

You will notice that my remarks have mainly been concerned with Geometry; but much had also to be done for Arithmetic and Algebra.

Arithmetic suffered from too long sums and from an excessive number of blind rules.

Algebra suffered because the elementary textbooks were arranged on a logical system as the subject might be arranged in the mind of a master of the subject. The elementary books started with long chapters on the four rules and other manipulation (which were meaningless to the child) before equations and problems were begun—the beginner could have little idea of the uses of algebra and so little interest in the work. Examination papers were largely concerned with manipulation. Graphs belonged only to science; they were almost, if not entirely, unknown in the mathematical classroom.

In spite of all that, there was little reason why the elementary teaching should not have been made more sensible at a much earlier date, but the fact remains that little was done in the matter of reform before the Teaching Committee came into being.

I think it is true to say that, at the beginning of this century mathematical textbooks were arranged on a logical system instead of a psychological one. Any new subject introduced to a child should attract his interest at once and should be related to experience the child already has had, many assumptions may be made, and of manipulation there should be a minimum—the logic of the subject belongs to a later stage.

It is interesting to note that in the early days of the A.I.G.T. members were requested to send in recommendations, syllabuses, etc. *in print* and they were to have very wide margins in which comments were to be made—this was done without any expense to the A.I.G.T.

Again in the early days of the Teaching Committee, apart from the cost of printing reports, the total expenses of the Teaching Committee amounted to

less than £2 a year. The only expenses were for postage; all the members paid their own travelling expenses, and consequently the Committee tended to be limited to members within fairly easy reach of London, which was a pity. All duplicating was done by the secretary or other members of the Committee. Travelling expenses were not paid by the Association at any rate till after the 1923 Geometry Report was published.

Today, what with travelling expenses, postage, typing and duplicating, the cost of each of our reports (apart from the printing) runs to anything from £150 to well over £300. This is a serious drain on our resources; the Committee and its sub-committees must try to economise. Actually this great cost is limiting the number of reports that the Association can afford to produce.

It is only fair to say that there is another side of the picture: the work of the Teaching Committee and its reports have done much to increase the membership of the Association; in 1898 there were just over 200 members, in 1950 there were over 2700.

I must pay tribute to some who had great influence in the early days of the Teaching Committee and made great contributions to the reforms effected.

First of all, I have already said that we owe a debt to Professor Perry for the publicity that he gave to the matter. To Professor Forsyth we owe much for the work he did as chairman of the B. A. Committee, as one who helped so much to get Cambridge and other examining bodies to dethrone Euclid and for the help he constantly gave to the M. A. Teaching Committee, particularly in the chair when he was president of the Association. At most of the early meetings of the Committee Alfred Lodge was in the chair and to him we owe a great debt: he was very wise and impartial and to me, as secretary, he was most helpful.

Perhaps the man who had the greatest influence on the Committee up till the time of his premature death in 1924 was Charles Godfrey: he was senior mathematical master at Winchester from 1899 till 1905, with a very free hand, in 1905 he became headmaster of Osborne and so had the ear of the Headmasters Conference which of course had great weight with examining bodies; he contributed many papers at the Annual Meetings of the M. A. and helped much with the drawing up of all our reports up to and including the Geometry report of 1923; he was one of the British representatives on the International Conference of Mathematicians; he was frequently consulted by examining bodies and for many years helped to set the papers for the Oxford and Cambridge School Certificate Examinations and other School Certificate Examinations.

From 1908 onwards the Committee was much influenced by Sir Percy Nunn: he made weighty contributions to the literature of teaching and trained many teachers. Then Miss Margaret Punnett, who worked so closely with Sir Percy Nunn, did much, especially for the beginnings of Arithmetic and for Girls' Schools and I believe she did all the typing and duplicating for the 1923 Geometry report. One other man I must mention is Mr. Tuckey: he has done much for the Committee, he was an original member of the Committee and has been an active member of it right up to the present day.

One personal note I may be allowed. In the very early days of the Committee there were two or three members who were writing textbooks. Godfrey and I used to be amused by them, trying to get decisions from the Committee on points that would help them in writing their books; I remember Godfrey remarking to me "At any rate we have no axes to grind—we are not writing textbooks." Little did we think at that time that six months later the Cambridge Press would ask us to write a book on Elementary Geometry—our first book.

A. W. S.

25-POINT GEOMETRY.

BY H. MARTYN CUNDY.

FINITE Galois arithmetics are well-known ; finite geometries however, though more interesting to the amateur, have not really acquired professional status and do not appear to any great extent in standard works. The following example arose from a chance remark in *Mathematics for T. C. Mits*, by L. R. and H. G. Lieber ; from this I deduce that a full theory has been worked out, but I have not seen it, and as far as I am concerned what follows is original, and I hope readers of the *Gazette* may find it new and stimulating.

1. *Basic structure.*

Consider the array of letters

$$\begin{array}{ccccc} A & B & C & D & E \\ F & G & H & I & J \\ K & L & M & N & O \\ P & Q & R & S & T \\ U & V & W & X & Y \end{array}$$

and denote by p the operation of replacing this array by

$$\begin{array}{ccccc} A & I & L & T & W \\ S & V & E & H & K \\ G & O & R & U & D \\ Y & C & F & N & Q \\ M & P & X & B & J. \end{array}$$

The reader can discover for himself the rule by which the transposition is effected. If the operation is repeated we obtain a new array, but a further repetition merely gives the first array with the order of rows and columns altered ; the first row reads $AEDCB$ and the first column $AUPKF$. The three arrays formed by the operations 1, p, p^2 are here set out in a table :

$$\begin{array}{ccccccccccccc} A & B & C & D & E & & A & I & L & T & W & & A & H & O & Q & X \\ F & G & H & I & J & \longrightarrow & S & V & E & H & K & \longrightarrow & N & P & W & E & G \\ K & L & M & N & O & p & G & O & R & U & D & p & V & D & F & M & T \\ P & Q & R & S & T & & Y & C & F & N & Q & & J & L & S & U & C \\ U & V & W & X & Y & & M & P & X & B & J & & R & Y & B & I & K. \end{array}$$

We regard the letters as *points* ; every row and column in any array as a *line* ; two rows or two columns in the *same* array are lines having no point in common and will be called *parallel* ; a row and column in the *same* array are *perpendicular*.

Distance.

The lines are regarded as closed and the points on them as cyclically permutable. The distance between two points is the shortest number of steps separating them on the line which joins them ; row-wise and column-wise distances are regarded as incommensurable. Thus we write

$$AB = AE = LT = DF = 1, \qquad AC = AD = LW = DM = 2,$$
$$AF = AM = 1', \qquad AK = AY = 2',$$

column-wise distances being denoted by dashed numerals. Sense is not taken into account at present, and there are no axioms of order. Note that

$$2 \times 1 = 2, \quad 2 \times 2 = 1, \quad 2 \times 1' = 2', \quad 2 \times 2' = 1'.$$

Axioms.

It is easy to verify the following axioms :

(*a*) There is one and only one line joining any two points.

(*b*) Two lines meet in one point unless they are parallel.

(*c*) Through any point there is one and only one line parallel to a given line.

(*d*) Through any point there is one and only one line perpendicular to a given line.

The geometry is therefore *plane* and partially metrical. The concept of angle cannot be developed satisfactorily in any manner which satisfies the fundamental congruence axiom (SAS).

2. *The fundamental transformation-group.*

I have already defined the operator p. Let us regard the identity operator as including any *cyclic* permutation of rows or columns or both ; that is, let us confine ourselves for the moment to transformations which keep one point, say A, fixed. Denote by i the operation of reversing the cyclic order in the rows. The operation of reversing the order of both rows and columns (if we like, of turning the plane through 180°) commutes with all other operations considered and is conveniently denoted by -1. These three operations preserve " distance " and " angle " and generate the group of *congruent rotations* about A. The group is of order 12, containing the elements ± 1, $\pm i$, $\pm p$, $\pm p^2$, $\pm ip$, $\pm ip^2$, which are connected by the relations

$$i^2 = 1, \quad p^3 = -1, \quad pi = -ip^2, \quad ip = -p^2 i.$$

(This is the *dihedral group* of the regular hexagon D_6, i being a reflexion in the y-axis, and p a rotation about the origin through $\frac{1}{3}\pi$.) The relations are easily verified by direct operations on the arrays of letters.

The only further operations which preserve right-angles and parallels, but not distance, are the elements of the product of this group with the operation q, defined as the operation of doubling distances in rows and interchanging rows and columns. (Neither of these operations separately preserves parallels when combined with p.)

Thus, the first array becomes

$$\begin{array}{ccccc} A & B & C & D & E \\ F & G & H & I & J \\ K & L & M & N & O \\ P & Q & R & S & T \\ U & V & W & X & Y \end{array} \quad \xrightarrow[q]{} \quad \begin{array}{ccccc} A & F & K & P & U \\ C & H & M & R & W \\ E & J & O & T & Y \\ B & G & L & Q & U \\ D & I & N & S & X \end{array}$$

The reader can now verify that $q^4 = -1$, $qi = -iq$, $pq = -qp^2$, $pq^2 = q^2 p$, $ip^2 q = pqi$. The extended group is of order 48, and includes all the *similar rotations* about A, since q obviously leaves *ratios of distances* unaltered. (Remember that $2 \times 2 = 1$ and that 1 and $1'$ are incommensurable.) These rotations carry B into any one of the 24 points other than A, combined with the ' reflection '' $-i$ about the line AB. This last operation transforms the right-angled triangle ABG into ABV, and the 24 " rotations " transform these into 24 similar pairs of right-angled triangles with one vertex at A and any one of the other 24 points at the right-angled corner, including of course the identical pair themselves.

If now we consider the 25 cyclic changes of rows and columns which carry A into any other point (the " pure translation "), including the identity, and form the product, we obtain 1200 similarity transformations of the configuration into itself, of which 300 are congruent transformations. The four operations 1, q, q^2, q^3 can be considered as " magnifications without rotation ".

To sum up, the congruence group of rotations is generated by the elements -1, i, p; and the similarity group is the product of this group by the cyclic group on the additional generator q. The operator q changes 1 to 2′, 2 to 1′, 1′ to 1, and 2′ to 2. Further, the full congruence transformation group is transitive on all the 25 points.

3. *Triangles and parallelograms.*

I shall not attempt here to develop the geometry logically from the minimum of axioms. It will be more interesting, I think, to indicate some of the methods of proof and to outline the results that can be obtained. It will be found that almost every euclidean theorem expressible in this geometry is true in it. To prove any particular result, we have only to verify it for a few cases which we can show are transformed into all other possible cases by the operations of the congruence or similarity groups, C and S. We begin with triangles. There are $25 \cdot 24 \cdot 20/3! = 2000$ of these, formed by any three non-collinear points. They are of three types only. 1200 are scalene right-angled triangles, obtained from ABF by the operations of S. The sides of ABF are $11'2'$ and the others are similar to it, of four " sizes ", found by magnifying ABF by $1, q, q^2, q^3$. A set of one of each size is ABF ($11'2'$), APB ($2'12$), ADP ($22'1'$) and AUD ($1'21$); (q removes B to the position of P; that is, AB becomes $AB' = AP = 2'$; operating on lengths, q is the cycle ($1'12'2$)). A further 200 are equilateral, for example, ABI, with four lengths of side. The remaining 600 are isosceles, similar to ABH ($112'$). The group S carries the representative triangles ABF, ABI, ABH into every other triangle; 6 operations of S carry ABI into itself, and 2 operations ABH, owing to their symmetry.

The important *midpoint theorem*, that the line joining the midpoints of the sides of a triangle is parallel to the base of the triangle and equal to half the base, can be verified by the reader in the three basic cases by the following figures; the theorem is invariant under S, therefore it is true for all cases.

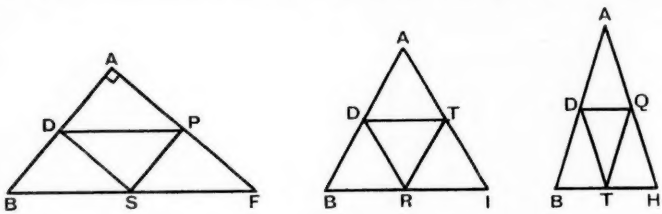


FIG. 1.

The fact that opposite sides of a parallelogram are equal follows from the observation that if we regard the 25 points as forming a rectangular array in a fundamental cell of a euclidean point-lattice (or on the universal covering-surface of a torus), the operations p and q preserve euclidean parallels. Hence a parallelogram in the finite geometry is a euclidean parallelogram on the lattice and its opposite sides are equal in both geometries. By the same argument its diagonals bisect one another.

4. *Circles.*

There are six points distant 1 from A, namely B, E, I, W, H, X. These lie on a *circle*. There will be 100 circles, with 25 centres and 4 radii. Since in the triangles ABI and ABH, AR is perpendicular to BI and AT to BH, the altitudes of all isosceles (and equilateral) triangles bisect the bases, so that the

centre of a circle through three non-collinear points can be found uniquely by the usual construction. It follows from the midpoint theorem that the angle in a semi-circle is a right angle, for instance, BIE. There is only one line through any point on the circle which does not meet it again, and it is perpendicular to the radius; for example, for the circle $BEIWHX$, the *tangent* at B is $BGLQV$, and it is perpendicular to AB.

The Simson line.

For the triangle BEI and the point H on its circumcircle, the feet of the perpendiculars are the collinear points QYC. The general investigation is left to the reader.

The isosceles triangle.

The triangle ABH has midpoints TQD, centroid L which divides the medians in the ratio 2 : 1, circumcentre I, feet of altitudes TOC, orthocentre W, Euler line $AILTW$, on which $WL = 2LI$. The "nine points centre" is A, and the "nine points circle" contains the points $TOCQDL$.

The equilateral triangle.

The triangle BWH has in addition an incentre A; its circumcircle is $BEIWHX$, also with centre A; the triangle IEX is congruent to BWH, and the triangles LDQ, OCT are congruent and equilateral.

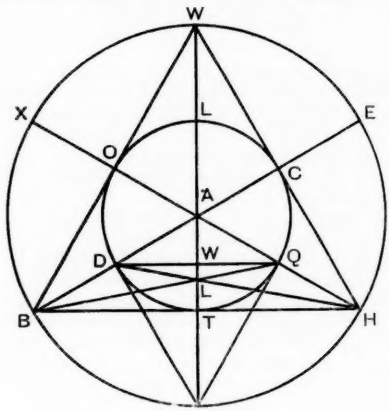


FIG. 2.

The right-angled triangle.

The properties are left to the reader.

5. *Polar properties and inversion.*

Consider the circle centre M and radius 1; it contains the six points $LNPJTF$. The remaining eighteen points fall into two classes: (i) points on tangents, $BCDGIKOQSVWX$, each on two tangents, and (ii) points which are midpoints of chords, $AEHRUY$. The polars of the points (i) can be identified as chords of contact of tangents; they are perpendicular to the lines joining the poles to the centre. For example, B lies on $BGLQV$, the tangent at L, and $OWFSB$, the tangent at F. Thus LF is the polar of B and is perpendicular to MB at X. For six of these points $BDKOVX$, the line joining

the point to the centre meets the circle in two points, and the polar is the perpendicular to this line through the fifth point on it. For the remaining six, and the points in class (ii), the lines joining them to the centre do not meet the circle, but the polars can be obtained by the reciprocal property. For example, A lies on the polar of O ($AFKPU$), and of D ($ANVJR$). Hence the polar of A is OD, which is perpendicular to AM, and parallel to the chord TAL with midpoint A. The reciprocal property can be shown to hold throughout.

Inverse points.

For the above circle, inverse points are the pairs

$$BDKAEHRUY, \qquad XVOGIWCQS.$$

Inspection shows that, if we now take sense into account, inversion in a circle of radius 1 is the transformation $1 \to 1$, $2 \to -2$, $1' \to -2'$, $2' \to -1'$. Inversions in circles of other radii are obtained by transforming these relations by the operations q. Since two circles do not necessarily intersect, the standard euclidean procedure cannot be carried through, but it will be found that a straight line inverts into a circle through the centre; thus $SVEHK$ invert into $YDIWO$, which lie on the circle $YDIWOM$, centre S and radius $2'$. Also a circle not through M inverts into a circle; for example, $GDISPT$, centre R and radius 2, inverts into $AVEXPT$, centre W and radius 1. The centres are not inverse.

6. *Parabolas.*

The locus of a point which moves so that its distance from A is equal to its distance from $CHMRW$ has five points on it, namely, $BIXTO$. There are 600 such parabolas, each occurring twice if the operations of S are applied to one of them. We have then only to verify results for this particular case. The following familiar results are seen to be true (Fig. 3):

$ABCDE$ is the axis, IT, XO are focal chords. The tangent at the vertex B is $BGLQV$. The tangent at I is $QEMUI$; the foot of the perpendicular from A to this is Q, on the tangent at B. The tangents at the ends of a focal chord IT meet at right angles at M on the directrix. AM is perpendicular to IT. The tangents at I, T, X are $QEMUI$, $VDFMT$, $LERFX$. They form a triangle MEF, whose orthocentre M lies on the directrix, and whose circumcircle $MWEFAJ$ passes through the focus. The chords IT, BX are parallel; their midpoints lie on a line LM parallel to the axis, which meets the curve at O and the directrix at M, where the tangents at I, T meet. The tangents at B, X meet at L, also on the line, and the tangent at O, $GORUD$, is parallel to the chords.

7. *Projective geometry.*

We now add to the rows in the first block an " infinity point " r_1, in which the " parallel lines " formed by the rows meet; and to the columns another " infinity point " c_1. We define r_2, c_2, r_3, c_3 similarly for the second and third blocks. If we consider r_1, r_2, r_3, c_1, c_2, c_3 to lie on a single line, we obtain a configuration of 31 points and 31 lines such that six lines pass through every point, six points lie on every line; every two lines without exception meet in a unique point, and every two points are joined by a unique line. In addition, the complete quadrangle construction leads to a unique harmonic conjugate. Fig. 4 shows three constructions for the harmonic conjugate of W with respect to C and M. Note that R, W; H, c_1 are both harmonic pairs with respect to C, M and similarly for all other ranges.

The axioms of projective geometry are therefore satisfied, and the projective theory of the conics can be developed. In particular, Pascal's theorem

is true and enables us quickly to obtain the six points which comprise a conic, given four or five of them. For example, through $AGQW$ we find, apart from

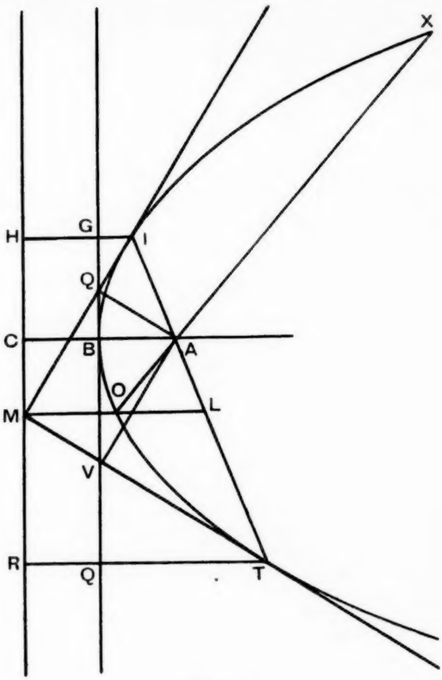


FIG. 3.

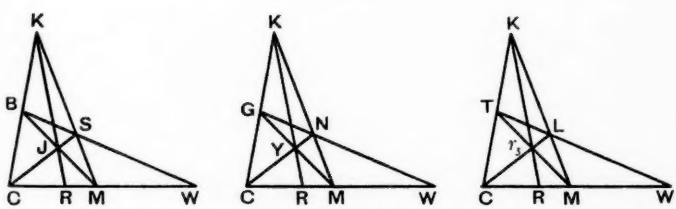


FIG. 4.

line-pairs, only three conics (Fig. 5, p. 164). In this case, because AW, GQ have a common midpoint L, though not in general, these three conics have the same centre, L. The first two can be called *ellipses* ; the first has two equal diameters and one different ; the second has all its diameters different. The third is a *hyperbola* ; the tangents at r_1, c_3 are $KLMNO$ and $HPDLY$, which meet at L, the centre.

In this case we began with four points not on a parabola or circle. If we take four points on a parabola, say $BIXT$, we again obtain three conics, namely,

an ellipse $BIXTEV$, with centre L,
an ellipse $BIXTFQ$, with centre M,
the parabola $BIXTOr_1$.

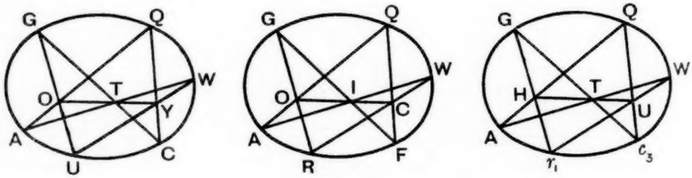


FIG. 5.

The centres (including those of the line-pairs) are the six points K, L, M, N, r_1, r_2, of which the first five lie on a line.

Similarly, if we begin with four concyclic points, for example, $HRGN$, we obtain

the circle $HRGNKQ$, with centre O,
the ellipse $HRGNSL$, with centre M,
the ellipse $HRGNXY$, with centre A.

The centres, and those of the line-pairs, are the six points O, M, A, W, J, U which, as is easily verified, lie on a conic. Note also that in each case the "new" six points, namely, $UCRFr_1c_3$, $EVFQOr_1$, $KQSLXY$, also lie on a conic. (The first is the hyperbola, centre L, asymptotes Lr_1, Lc_3; the second the parabola focus H and directrix $DINSX$; the third an ellipse centre R.) We conclude then that through any four points three conics can be drawn, excluding line-pairs, each containing two other points. These extra six points themselves lie on a conic. The centres of the three conics and the three line-pairs lie on a further conic. Of course, there is one and only one conic through five points, no three of which are collinear.

8. *The rectangular hyperbola.*

The conic through G touching Ar_1, Ac_1 at r_1, c_1 is $RNGYr_1c_1$. This is a rectangular hyperbola, centre A; RAN, GAY are diameters. The tangent at G is $XGTCK$; it meets the asymptotes at C, K and $CG=GK$. All the triangles formed by three points on the curve are right-angled, so that the orthocentric property has no significance. In fact, $NGRY$ is a parallelogram in which each diagonal is perpendicular to a pair of opposite sides (Fig. 6).

9. *Conclusion.*

I have said enough to indicate the very large scope and some of the fascination of this geometry. I have not investigated at all its many peculiar properties in which it differs from euclidean geometry, but it is amusing to see all the familiar results coming out. What more will you have? Why bother about a continuum when 25 points will do all the tricks? Or are there really only 25 points? In considering the parallelogram we had recourse to an infinite lattice. This approach suggests a euclidean model for the geometry. Suppose the length 1 is called k, and $1'=l$. Then if $l=k/\sqrt{3}$, the operation $-p$ is a plane rotation of the rectangular euclidean lattice, through $2\pi/3$, with a reduction of all distances, modulo $5k$ or $5l$. (It is then clear why the group

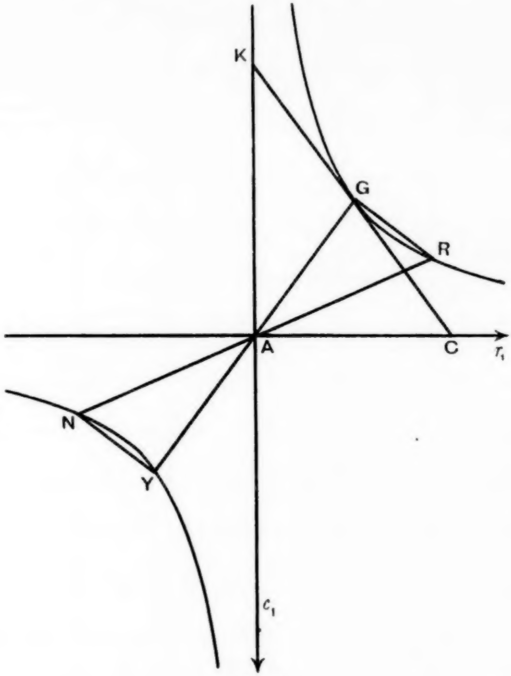


FIG. 6.

generated by $\pm 1, p, i$ is the dihedral group D_6). This is apparent from Fig. 7, in which $AB \equiv k$, $AF \equiv l$, $AI \equiv -k$ (mod. $5k$ or $5l$) and $\angle BAI = 2\pi/3$.

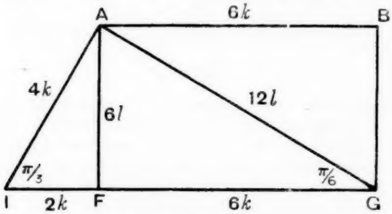


FIG. 7.

p carries AI to AB, and AS to AF.

If this rotation is followed by a reflection in the centre A (the operation -1), reversing the sign of k, it is seen to be equivalent to p. It is easier to see the rotation on the lattice (Fig. 8); p is a rotation in the clockwise direction through $\pi/3$. The operation qi in this model is now seen to be a magnification by $2/\sqrt{3}$ coupled with a rotation through $\frac{1}{2}\pi$; AB is carried to AP,

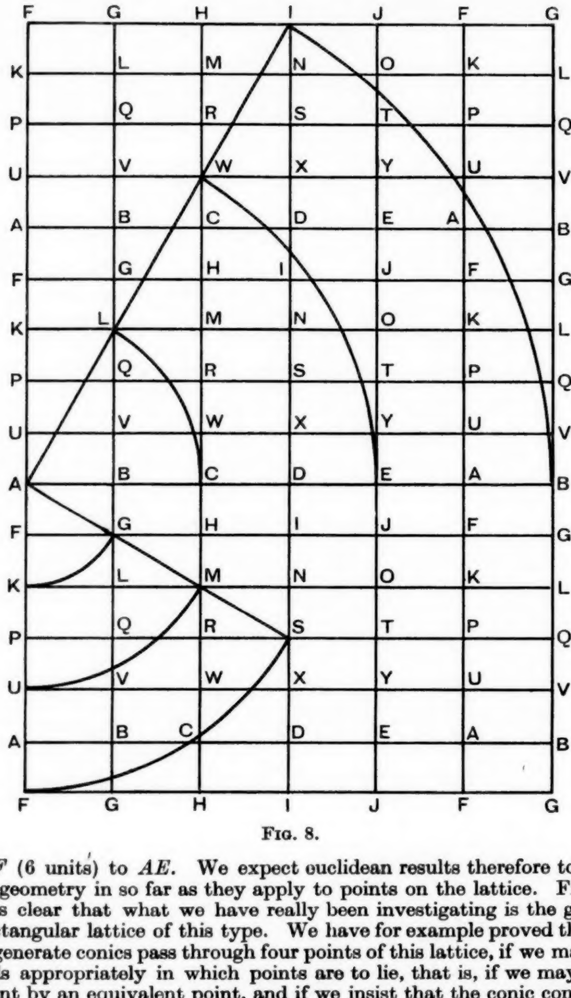


FIG. 8.

and AF (6 units) to AE. We expect euclidean results therefore to be true in this geometry in so far as they apply to points on the lattice. Finally, it becomes clear that what we have really been investigating is the geometry of a rectangular lattice of this type. We have for example proved that three non-degenerate conics pass through four points of this lattice, if we may select the cells appropriately in which points are to lie, that is, if we may replace any point by an equivalent point, and if we insist that the conic contains an additional lattice-point. Further every conic through five points of the lattice contains a sixth, possibly at infinity, provided equivalent points are suitably selected. If we remove this restriction, the result provides that every conic through five points of the lattice contains another *rational* point of the lattice; that is, any conic through five points with rational coordinates contains a sixth such point; but this is obvious anyway by Pascal's theorem.

H. M. C.

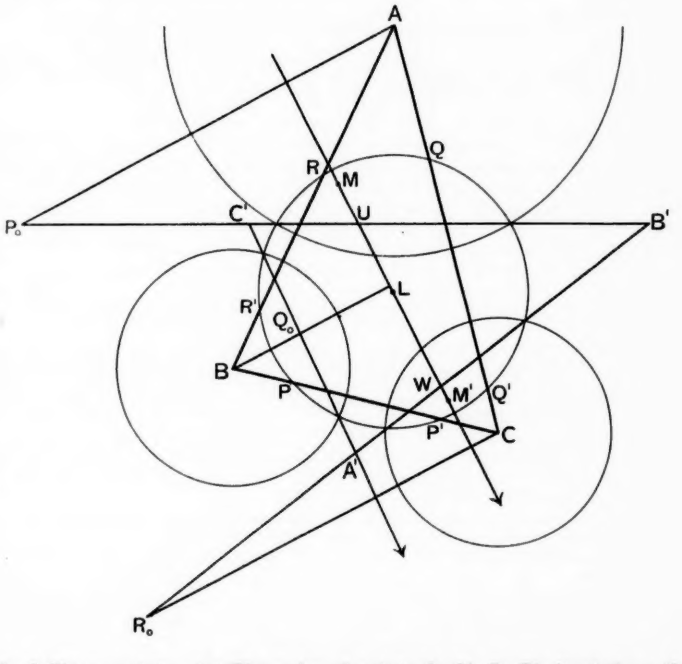
ISOGONAL CONJUGATE POINTS FOR A TRIANGLE.

BY N. A. COURT.

1. *Introduction.* (*a*) Let M, M' be two isogonal conjugate points with respect to a triangle $(T) = ABC$: MP, $M'P'$; MQ, $M'Q'$; MR, $M'R'$ the perpendiculars from M, M' upon the sides BC, CA, AB of (T), respectively. The triangles $(M) = PQR$, $(M') = P'Q'R'$ are the pedal triangles of the points M, M' for (T), and the common circumcircle (L) of those two triangles is the pedal circle of M. M' for (T) [**1**, pp. 238–242].

We shall denote by (A), (B), (C) the circles having A, B, C for centres and orthogonal to the circle (L).

(*b*) *Definition.* The triangle $(M_0) = (P_0Q_0R_0)$ having for vertices the points $P_0 = (QR, Q'R')$, $Q_0 = (RP, R'P')$, $R_0 = (PQ, P'Q')$ shall be referred to as the copedal triangle of the points M, M' for the triangle (T).



2. *Collinear points.* (*a*) The pairs of points Q, Q', R, R' situated on the circle (L) and collinear with the centre A of the circle (A), orthogonal to (L), are pairs of inverse points with respect to (A), hence the perpendiculars QM, RM to the lines AQQ', ARR' are the polars of the points Q', R' with respect to the circle (A). Similarly, the lines $Q'M'$, $R'M'$ are the polars of the points Q, R for (A). Hence the points M, M' are the poles of the lines $Q'R'$, QR for (A).

Analogous considerations apply to the circles (B), (C). Thus : *If M, M' are two isogonal conjugate points for a triangle (T), the sides of the pedal triangles of*

M', M for (T) are the polars of the points M, M' for the circles having for centres the vertices of (T) and orthogonal to the pedal circle of M, M' for (T), respectively.

(b) It follows from the preceding considerations that the two triangles MQR, $M'R'Q'$ are polar reciprocal with respect to the circle (A), therefore the two triangles are homological, and the three lines MM', QR', $Q'R$ are concurrent [2 and 3]. Hence the point $U=(QR', Q'R)$ lies on the line MM'. Thus: *If M, M' are two isogonal conjugate points for an angle $(AQR, AQ'R')$, and QR, $Q'R'$ are the pedal lines of M, M' for that angle, the point $U=(QR', Q'R)$ lies on the line MM'.*

(c) Considering the circles (B), (C) it may be shown in an analogous way that the points $V=(RP', R'P)$, $W=(PQ', P'Q)$ also lie on MM'. Thus: *If M, M' are a pair of isogonal conjugate points for a triangle (T) and PQR, $P'Q'R'$ the pedal triangles of M, M' for (T), the three points*

$$U=(QR', Q'R), \quad V=(RP', R'P), \quad W=(PQ', P'Q)$$

are collinear.

The line UVW joining the three points passes both through M and M'.

Observe that the first part of this proposition may be obtained by applying Pascal's theorem to the simple hexagon $PQ'RP'QR'$ inscribed in the circle (L). The line MM' is thus the Pascal line of this hexagon.

3. *The copedal triangle.* (a) The vertex $P_0=(QR, Q'R')$ of the copedal triangle $(M_0)=P_0Q_0R_0$ is the point of intersection of the polars QR, $Q'R'$ of the points M', M for the circle (A) (art. 2a). Similarly for the vertices Q_0, R_0 of the triangle (M_0) with regard to the circles (B), (C), respectively. Thus the lines AP_0, BQ_0, CR_0 are all three perpendicular to the line MM', and therefore parallel. Thus: *If M, M' are a pair of conjugate isogonal points for a triangle (T), the copedal triangle of M, M' for (T) is perspective to (T). The centre of perspectivity is the point at infinity in the direction perpendicular to the line MM'.*

The axis of homology is thus an axis of affinity of the two triangles (T) and (M_0).

(b) The points $P_0=(QR, Q'R')$, $U=(QR', Q'R)$, $A=(QQ', RR')$ are the vertices of the diagonal triangle AUP_0 of the complete quadrangle $QQ'RR'$ inscribed in the pedal circle (L) of M, M' for (T), hence the lines AP_0, AU are harmonically separated by the sides $AQQ'C$, $ARR'B$ of (T) and the points P_0, U lie on the polar of A for the circle (L). Consequently the poles B', C' of the lines AC, AB with respect to (L) are collinear with and harmonically separated by the points P_0, U. Thus, if A' is the pole of the line BC with respect to (L), we have the harmonic pencil of lines $A'\ (B'C', P_0U)$.

Considering the vertices B, C of (T) we obtain in an analogous way the harmonic pencils of lines $B'\ (C'A', Q_0V)$, $C'\ (A'B', R_0W)$.

Now the three points U, V, W are collinear (art. 2c), hence the three cevians $A'P_0$, $B'Q_0$, $C'R_0$, of the triangle $A'B'C'$ meet in a point, K. Thus: *If M, M' are a pair of isogonal conjugate points for a triangle (T), the copedal triangle (M_0) of M, M' for (T) is inscribed in and perspective to the polar reciprocal triangle (T') of (T) with respect to the pedal circle of M, M' for (T). The axis of perspectivity of the two triangles (M_0), (T') is the line MM'.*

4. *A trilinear polar.* (a) The lines AP_0, AU are separated harmonically by the sides AB, AC of (T) (art. 3b), hence the trace U' of AU on the side BC of (T) is separated harmonically by B, C from the trace U'' of AP_0. Similarly, for the pairs of lines BQ_0, BV; CR_0, CW, with regard to the sides CA, AB of (T), respectively.

Now the three lines AP_0, BQ_0, CR_0 are parallel (art. 3a), that is they have in common a point I at infinity, in the direction perpendicular to the line MM',

hence the traces U', V', W' of the lines AU, BV, CW on the sides BC, CA, AB are collinear, and the line joining them is the trilinear polar of the point I for the triangle (T). Thus : *If the points M, M' are isogonal conjugates for a triangle (T), the lines joining the vertices of (T) to the points U, V, W (art. 2c), respectively, meet the opposite sides of (T) in three collinear points. The line joining the three points is the trilinear polar, for (T), of the point at infinity in the direction perpendicular to the line MM'.*

(*b*) The relation just considered (art. 4*a*) may be formulated in a different way, as follows : *If M, M' are a pair of isogonal conjugate points for a triangle (T), this triangle is inscribed in and perspective to the triangle formed by the lines AU, BV, CW (art. 2c). The centre of perspectivity is the point at infinity in the direction perpendicular to the line MM'.*

The vertices of the circumscribing triangle (AU, BV, CW) are the harmonic associates of the point I for the triangle (T) [**1**, p. 221].

(*c*) The last two propositions (arts. 4*a*, 4*b*) may be arrived at in a different manner.

The lines AU, BV, CW are the polars of the points P_0, Q_0, R_0 with respect to the pedal circle (L) (art. 3*b*). Thus the triangle (AU, BV, CW) is the polar reciprocal of the triangle (M_0) for the circle (L).

Now the triangle (M_0) is inscribed in and perspective to the triangle (T') (art. 3*b*), hence the polar reciprocal triangle (AU, BV, CW) of (M_0) with respect to (L) is circumscribed about the polar reciprocal (T) of (T') for (L), and the two triangles (AU, BV, CW), (T) are perspective.

Moreover, the axis and centre of perspectivity of the triangles (AU, BV, CW) and (T) are the polar and pole, for (L), of the centre K and the axis of perspectivity MM' of the triangles (M_0), (T'). Now the pole of MM' for (L) is the point at infinity I in the direction perpendicular to MM'. We thus draw the conclusion that the polar of K for (L) is the line $U'V'W'$, a relation that may readily be verified in a more direct way.

5. *Perspective triangles.* (*a*) Let M_a, M_a' be the traces of the lines AM, AM' on the lines QR, $Q'R'$, respectively. The lines AM, AM' are perpendicular to the lines $Q'R'$, QR, respectively, hence the latter two lines are two altitudes of the triangle AM_aM_a', issued from the vertices M_a', M_a of the triangle. Thus the point $P_0 = (QR, Q'R')$ is the orthocentre of the triangle AM_aM_a', and the third altitude AP_0 is perpendicular to the side M_aM_a' of the triangle.

Now, the line MM' is also perpendicular to the line AP_0 (art. 3*a*), hence the lines MM', M_aM_a' are parallel. Thus : *If M, M' are a pair of isogonal conjugate points for an angle A, the lines AM, AM' meet the pedal lines of M, M' for A in two points such that the line joining them is parallel to the line MM'.*

(*b*) Considering the vertices B and C of (T) we may obtain, in a similar way, lines M_bM_b' and M_cM_c' analogous to the line M_aM_a' obtained in relation to the vertex A, and the two lines would be parallel to the line MM'. Hence the lines joining corresponding vertices of the two triangles $M_aM_bM_c$, $M_a'M_b'M_c'$ are parallel. Thus : *If M, M' are a pair of isogonal conjugate points for a triangle (T), the lines joining M to the vertices of (T) meet the corresponding sides of the pedal triangle of M for (T) in the vertices of a triangle which is homological to the analogous triangle of the point M' for (T). The centre of homology of the two triangles is the point at infinity of the line MM'.*

The axis of homology is thus an axis of affinity of the two triangles.

6. *Applications.* The preceding properties of isogonal points of a triangle may be applied to the known pairs of isogonal points associated with a triangle, as the circumcentre and the orthocentre, the centroid and the Lemoine point, the Brocard points. This approach may yield properties that have already been obtained by other means, and perhaps others, worthy of mention, that

have escaped notice, so far. These applications are left for the entertainment of the interested reader.

University of Oklahoma. N. A.-C.

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2. Michel Chasles, *Sections coniques*, p. 98 (Paris, 1865).
3. L. Cremona, *Projective Geometry*, p. 239 (London, 1913).

CORRESPONDENCE.

To the Editor of the *Mathematical Gazette*.

DEAR SIR,—There are certain points which we should like to clarify to Mr. Inman, who reviewed *Mathematics Today Part I* in the February 1952 issue of the *Gazette*.

The material for *Mathematics Today* was collected in response to our own need for a textbook for the Alternative Syllabus. When we were persuaded that our own exercises might be of value to other teachers, we had to decide in which part of the course the need for exercises was most urgent since writing and publishing would be a lengthy process. Clearly this was the last three years of the five-year course where most of the new material would be introduced. Hence we had to decide which elementary processes we could assume to be taught universally in the first two years. We chose such straightforward topics as would be found in any elementary textbooks already in schools. (In these days of economy it was not to be considered that schools would equip themselves with new textbooks throughout at the same time.) Some topics, however, presented difficulty : graphs was one of these, for the treatment of this subject varies so widely. Here a detailed logical development of the subject seemed essential, not only because we endeavoured to make each chapter complete in itself, but also because we needed to extend this subject in Parts II and III. We hoped, too, that our elementary statistical graphs would prove of value during the first two years of the course. No teacher would take Chapter IV in its entirety : this is no watertight compartment, but a river which flows into innumerable mathematical channels to be introduced at the discretion of the teacher. There were other topics for which we found by experience that frequent revision is necessary. Hence examples on formulation, rectangular areas and volumes and percentages were included (the latter Mr. Inman appears to accept).

On the adequacy of revision papers, papers 10–15 each consist of three 40 minute tests arranged in the familiar Section A and B formation. Thus there are twenty-seven 40 minute tests and one 2 hour test paper. With a time allowance of 5 (and sometimes 4) periods a week for Mathematics, there is not time for more than one test paper per fortnight. Since also there are two parallel exercises on each topic, it was envisaged that the intelligent teacher would use the second set for revision purposes : thus there should be ample revision exercises.

Yours, etc., E. E. BIGGS and H. E. VIDAL.

GLEANINGS FAR AND NEAR.

1701. When these 400 girls meet their Headmistress they face a woman in a thousand.—From a school prospectus. [Per Mr. B. M. Brown.]

THE CONIC AND THE AUXILIARY CIRCLE.

By B. E. Lawrence.

1. Some years ago, under the heading " Introductory theorems in geometrical conics " (*Gazette*, XVIII (1934), p. 223), an elementary geometrical treatment was offered of some general properties of conics based on the focus-directrix definition

$$SP = e \,.\, PM. \quad \text{.....................................(i)}$$

Other definitions of the conic were not considered nor was the relation of the curve to its auxiliary circle introduced.

In coordinate geometry this circle enters readily as the locus of the foot of the perpendicular from the focus on the tangent, and the same approach is used here, starting from the fundamental property of the tangent, that the lines from its intersections with the curve and the directrix to the focus are perpendicular.

2. *The auxiliary circle.*

Draw the perpendiculars SX, ST to the directrix and the tangent to form a cyclic quadrilateral $STZX$.

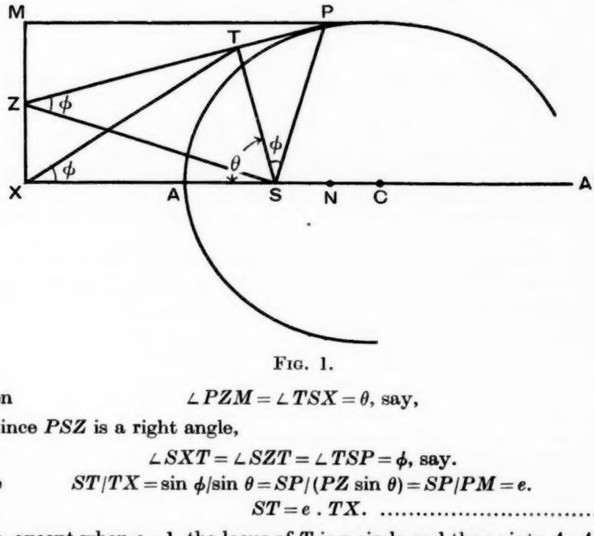

Fig. 1.

Then $$\angle PZM = \angle TSX = \theta, \text{ say,}$$

and, since PSZ is a right angle,

$$\angle SXT = \angle SZT = \angle TSP = \phi, \text{ say.}$$

Hence $$ST/TX = \sin\phi/\sin\theta = SP/(PZ\sin\theta) = SP/PM = e.$$

Thus $$ST = e \,.\, TX. \quad \text{.....................................(ii)}$$

Hence, except when $e = 1$, the locus of T is a circle and the points A, A' which divide SX internally and externally in the ratio $e : 1$ lie both on the circle (ii) and the conic (i). The line AA' is an axis of symmetry whose midpoint C is the centre of the circle.

It is clear that XZ is the polar of S, and it follows that the triangles CST, CTX are similar.

3. *The ellipse as the orthogonal projection of a circle.*

The similarity of (i) and (ii) suggests a relation between the curves they define. In fact, when the eccentricity is less than unity, it is easy to deduce

the locus of (i) from that of (ii) by orthogonal projection through an angle α whose sine is e.

Suppose we have a circle on which the distances of any point Q from fixed points S, X satisfy the relation

$$SQ = e \,.\, QX.$$

Project orthogonally on to a plane through SX at an angle α. Let P be the

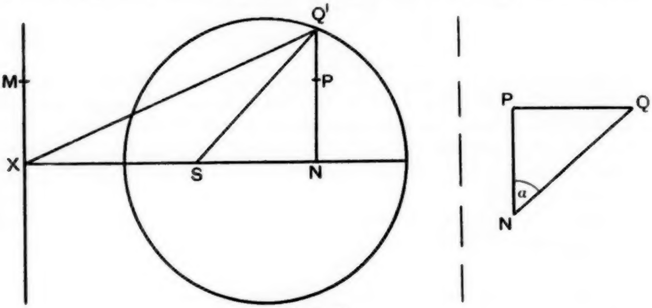

FIG. 2.

projection of Q, and XNQ' the position XNQ takes up when rotated through the same angle α.

Then $\qquad PQ/SQ = QN \sin \alpha / e \,.\, QX = QN/QX = Q'N/Q'X.$

Hence, the right-angled triangles SPQ, XNQ' are similar and so

$$SP/PM = SP/XN = SQ/XQ' = SQ'/XQ' = e.$$

Therefore, the orthogonal projection of the circle is the locus (i), that is, an ellipse.

4. As an illustration of an alternative method we can show that *the orthogonal projection of the locus* $SP = e \,.\, PM$ *on a plane through its directrix at an angle* $\sin^{-1} e$ *is a circle.*

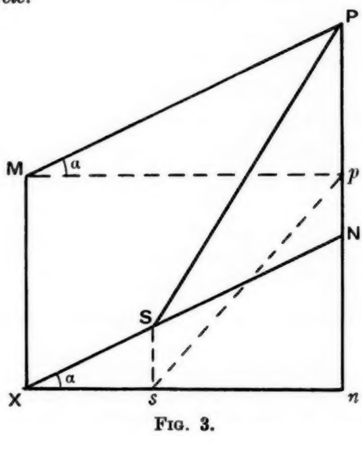

FIG. 3.

Let p, s, n be the orthogonal projections of P, S, N.

Then
$$\begin{aligned} pX^2 &= pM^2 + MX^2 \\ &= PM^2 \cos^2 \alpha + PN^2 \\ &= PM^2(1-e^2) + SP^2 - SN^2 \\ &= PM^2 - SN^2. \end{aligned}$$

Hence
$$\begin{aligned} e^2 \,.\, pX^2 &= e^2 \,.\, PM^2 - e^2 \,.\, SN^2 \\ &= SP^2 - (SN \sin \alpha)^2 \\ &= SP^2 - (Nn - Ss)^2 \\ &= SP^2 - (Pp - Ss)^2 \\ &= sp^2. \end{aligned}$$

Thus,
$$sp = e \,.\, pX.$$

Hence, p describes a circle as stated.

There is an obvious distinction between ϵ the eccentricity of the conic, which is fixed for any particular conic, and e used in the circle locus, which is the ratio of the distance of an arbitrary point from the centre to the radius. By varying the arbitrary point or by projecting through different angles, we get a variety of conics from one circle. From the ellipse by orthogonal projection we can derive a circle only by using $\sin^{-1} e$ as the angle of projection.

5. We can use the focus-directrix and orthogonal projection definitions as may be convenient for deducing the properties of the ellipse. The usual deductions of the constant ratio of ordinates in the circle and ellipse and of central properties including conjugate diameters can be made at once.

The auxiliary circle can be used in reciprocation, and the circle itself reciprocated into the ellipse from which it was derived. However, the usual proof, using Salmon's theorem, that the reciprocal of a circle is a conic is brief and gives the three types of conic.

The converse proposition that the reciprocal of a conic with respect to a focus is a circle is readily proved.

6. *Polar reciprocal of a conic.*

The reciprocal of a conic for a given circle K is the locus of the poles (with respect to K) of tangents to the conic. Hence, in Fig. 4 (p. 174) we take as a typical pole a point t on ST such that, if k is the radius of K,

$$ST \,.\, St = k^2.$$

Now, the locus of T is the auxiliary circle and the inverse of T is t. But the inverse of a circle is a circle. Hence t describes a circle which is the inverse of the auxiliary circle with respect to K. To get a little more information about the reciprocal and, incidentally, to avoid quoting the inversion theorem, we may proceed thus: take the fixed point x on SX so that

$$Sx \,.\, SX = St \,.\, ST = k^2.$$

Then the triangles Stx and SXT are similar, and so, by (ii),

$$Sx/xt = ST/TX = e.$$

Therefore
$$xt = Sx/e = k^2/e \,.\, SX = k^2/l.$$

Thus the locus of t is a circle whose centre is x and radius k^2/l.

7. *Pedal equations of ellipse and hyperbola.*

We have, in Fig. 1,

$$CS^2 = CT^2 + ST^2 - 2CT \,.\, ST \cos CTS$$

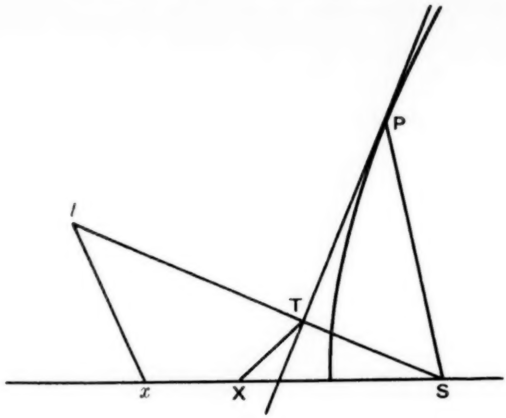

FIG. 4.

and since the triangles CTS, CXT are similar,

$$\cos CTS = \cos CXT = \cos \phi = p/r.$$

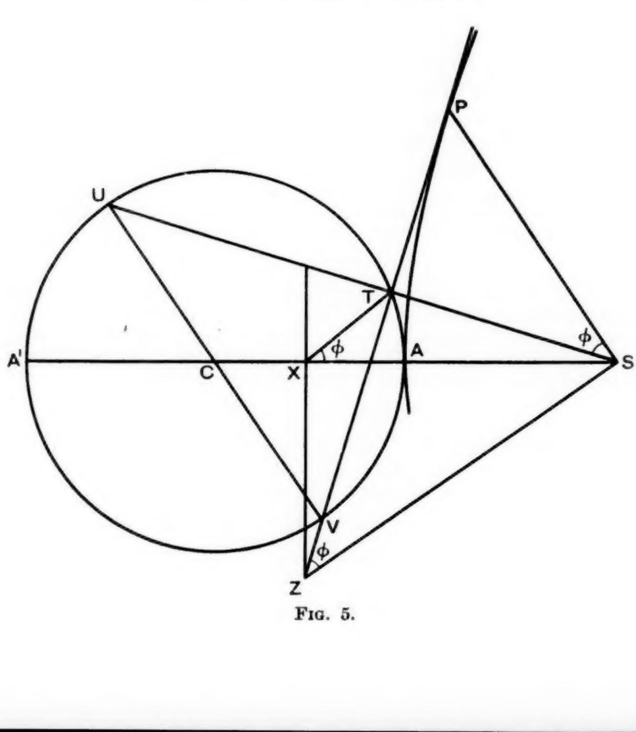

FIG. 5.

Hence $$a^2e^2 = a^2 + p^2 - 2ap^2/r,$$

or $$\frac{b^2}{p^2} = \frac{2a}{r} - 1.$$

We may illustrate a different method for the pedal equation of the hyperbola.

Let ST, PT cut the circle again in U, V and let $SU = p'$. Then (Fig. 5)

$$ST \,.\, SU = SA \,.\, SA'$$

or $$pp' = b^2.$$

Also, since UV is a diameter, and since CTS, CXT are similar triangles,

$$\begin{aligned} \angle CUT = \angle CTU &= \pi - \angle CTS \\ &= \pi - \angle CXT \\ &= \phi, \end{aligned}$$

that is, $$\angle CUT = \angle TSP,$$

and these are alternate angles. Hence, from the parallels UV, PS, we have

$$UV/SP = TU/TS$$

or $$2a/r = (p' - p)/p,$$

that is, $$\frac{2a}{r} = \frac{b^2}{p^2} - 1.$$

B. E. L.

1702. Why, in the light of the normal distances involved, is a shot-putt of 12 ft. 10¾ ins. awarded 25 points while a hammer throw of 12 ft. 9½ ins. is deemed to be worth 26 points? I suspect the reason is, that to obtain values right down from 1360 points to 1 point the rigid logarithmic curve, which made sense at mid-values, had to be followed to its bitter end. This appears to be the classic indictment against the rigidity imparted by a slavish adherence to a second-degree curve.—*The Athlete* (Spring, 1950), p. 31. [Per Mr. H. ApSimon.]

1703. An often quoted but rarely documented tale of π is that of the attempt to determine its value by legislation. House Bill No. 246, Indiana State Legislature, 1897, was written by Edwin J. Goodwin, M.D., of Solitude, Posey County. It begins as follows : " A bill for an act introducing a new mathematical truth and offered as a contribution to education to be used only by the State of Indiana free of cost by paying any royalties whatever on the same. . . .

Section I. Be it enacted by the General Assembly of the State of Indiana : It has been found that a circular area is to the square on a line equal to the quadrant of the circumference, as the area of an equilateral rectangle is to the square on one side. . . ."

The bill was referred first to the House Committee on Canals and then to the Committee on Education which recommended its passage. It was passed and sent to the Senate where it was referred to the Committee on Temperance which recommended its passage. In the meantime the bill had become known and ridiculed in various newspapers. This resulted in the Senate's finally postponing indefinitely its further consideration in spite of the backing of the State Superintendent of Public Instruction, who was anxious to assure his state textbooks of the use, free, of this copyrighted discovery.—Phillip S. Jones, *Mathematics Teacher*, March 1950, p. 122. A reference is given to *Proceedings of the Indiana Academy of Science*, 45 (1935), pp. 206–210.

CURVES OF CONSTANT DIAMETER.

By M. J. Kearsley.

The literature dealing with curves of constant diameter is not very extensive, and the following notes may be of general interest.

If the distance between parallel tangents to any curve is constant, whatever their common direction, then the curve is a curve of constant diameter. Thus, if $p(\psi)$ is the perpendicular distance from the origin to the tangent to the curve, where ψ is the angle the tangent makes with the positive x-axis, and $D(\psi)$ is the diameter of the curve, that is, the distance between parallel tangents, then

$$D(\psi) = p(\psi) + p(\psi + \pi).$$

The simplest example of such a curve is the circle, for which

$$p(\psi) = a + \lambda \sin \psi + \mu \cos \psi,$$
$$p(\psi + \pi) = a - \lambda \sin \psi - \mu \cos \psi,$$

where a is the radius, and λ, μ are constants, so that

$$D(\psi) = 2a, \quad \text{a constant.}$$

The existence of curves of constant diameter other than circles has been known for some time. They are mentioned in Liouville's *Journal de Mathématiques* (1860), p. 283, by Barbier, who gives Puiseux' example: the major axis of an ellipse divides the ellipse into two equal parts; if normals of length equal to the major axis are constructed on one half of the ellipse, the locus of their end points, together with the original half of the ellipse, form a continuous curve of constant diameter equal to the major axis of the ellipse (Fig. 1).

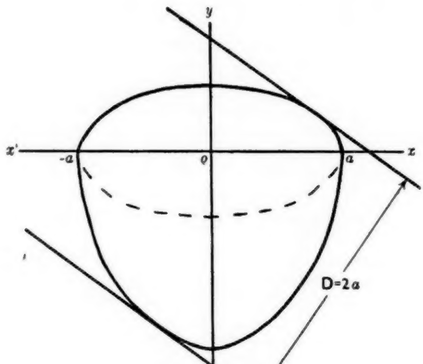

Fig. 1.

Another curve of constant diameter can be constructed from an equilateral triangle. With each vertex in turn as centre an arc of a circle of radius equal to the side of the triangle is drawn (Fig. 2). Since the curve does not

have a continuously turning tangent, the line through A parallel to the tangent at B, say, is considered as the tangent parallel to the tangent at B.

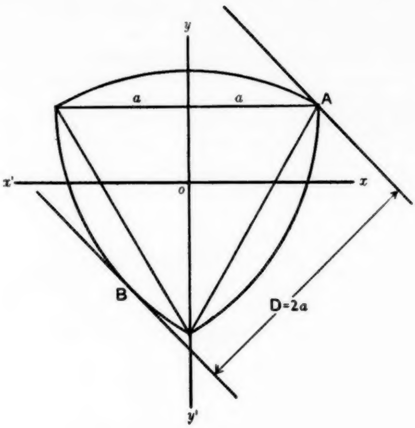

FIG. 2.

Curves of constant diameter of the same type can be constructed from any odd-sided regular polygon, by taking each vertex as centre and drawing arcs to pass through two opposite vertices.

The equations to a wide class of curves of constant diameter can be obtained from the Fourier series

$$p=a+\Sigma(A_n\cos n\psi+B_n\sin n\psi), \quad \text{.......................(i)}$$

provided that n is an odd integer, since then

$$p(\psi+\pi)=a+\Sigma(-A_n\cos n\psi-B_n\sin n\psi),$$

and hence

$$D=p(\psi)+p(\psi+\pi)=2a, \quad \text{a constant.}$$

The equation to the curve of constant diameter formed from the equilateral triangle (see Fig. 2) may be obtained in this form. If the centre of the triangle be taken as origin, and the axes are as shown, the equation is of the form

$$p=a+\sum_1^\infty a_n\cos 3n\psi,$$

where n is an odd integer and $2a$ is the side of the triangle, and

$$\tfrac{1}{3}\pi a_n=\{12na/(9n^2-1)\}\sin\tfrac{1}{2}n\pi+(4a/3n)\cos 3\pi n\sin\tfrac{1}{2}n\pi$$

so that

$$p=a+\frac{a}{2\pi}\cos 3\psi-\frac{a}{60\pi}\cos 9\psi+\frac{a}{280\pi}\cos 15\psi+\dots .$$

Also the equation to a curve of constant diameter $2a$ formed from any m-sided regular polygon (m odd) may be obtained in the form

$$p=a+\sum_{n=1}^{\infty}a_{nm}\cos mn\psi,$$

where m and n are odd integers and

$$\frac{1}{m}\pi a_{mn}=\frac{4amn}{(m^2n^2-1)}\sin\tfrac{1}{2}n\pi+\frac{4a}{mn}\cos mn\pi\sin\tfrac{1}{2}n\pi.$$

Ovals of constant diameter.

Any oval that can be expressed in form (i), where n is an odd integer, is a curve of constant diameter, but the converse is not necessarily true.

Consider, for example, the curve $p=1+\lambda\cos 3\psi$.
For $\lambda=1$, the curve crosses itself and so is not an oval. For $1>\lambda>\frac{1}{8}$, the curve is as in Fig. 3, and as $\lambda\to\frac{1}{8}$ the areas such as B tend to zero. At $\lambda=\frac{1}{8}$, the cusps

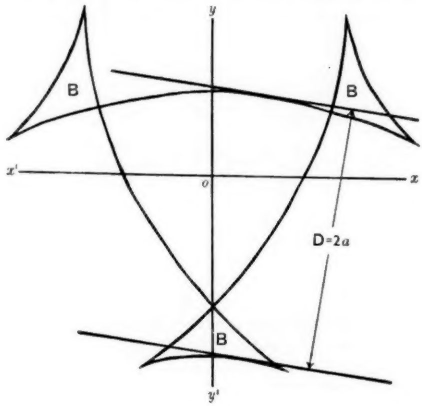

FIG. 3.

and nodes just disappear and the curve appears as in Fig. 4, while for $\frac{1}{8}>\lambda>0$ as $\lambda\to 0$ the curve tends towards a circle.

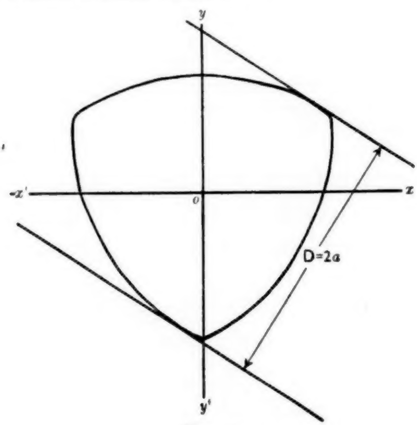

FIG. 4.

Next consider a more general curve of constant diameter two units:

$$p = 1 + \lambda \sin n\psi + \mu \cos n\psi, \qquad \text{(ii)}$$

where n is an odd integer. For this equation to represent an oval, the radius of curvature must be positive for all ψ. For the oval the radius of curvature is $p + p''$, where $p' = dp/d\psi$, $p'' = d^2p/d\psi^2$.

The radius of curvature of the curve (ii) is then

$$p + p'' = 1 - (n^2 - 1) \,.\, \sqrt{(\lambda^2 + \mu^2)} \,.\, \sin(n\psi + \alpha),$$

where $\tan \alpha = \mu/\lambda$. Since the radius of curvature is to be positive and the maximum value of $\sin(n\psi + \alpha)$ is 1, we must have

$$1 - (n^2 - 1) \,.\, \sqrt{(\lambda^2 + \mu^2)} \geqslant 0,$$

or, with $n > 1$,

$$\sqrt{(\lambda^2 + \mu^2)} \leqslant 1/(n^2 - 1). \qquad \text{(iii)}$$

Areas of ovals of constant diameter.

The perimeter of all ovals of constant diameter $2a$ is $2\pi a$, but the area of the oval depends on the values of the constants λ, μ, n, etc.

Again, consider curve (ii) subject to condition (iii). The area A of the oval is given by

$$A = \tfrac{1}{2}\int_0^{2\pi} p\, ds = \tfrac{1}{2}\int_0^{2\pi} (p^2 - p'^2)\, d\psi. \qquad \text{(iv)}$$

Substituting for p and p' in (iv) and integrating with respect to ψ from $\psi = 0$ to $\psi = 2\pi$,

$$A = \pi\{1 + \tfrac{1}{2}(\lambda^2 + \mu^2)(1 - n^2)\}.$$

Thus, for the oval of constant diameter of form (ii) to have least area, $n > 1$, since $n = 1$ gives a circle having maximum area. Also $\lambda^2 + \mu^2$ must have the maximum value, so that from (iii)

$$\sqrt{(\lambda^2 + \mu^2)} = 1/(n^2 - 1).$$

Hence

$$A = \pi\left\{1 - \frac{1}{2(n^2 - 1)}\right\}.$$

Now n must be an odd integer greater than 1. Thus $n = 3$ gives the minimum area. In this case,

$$A = 15\pi/16,$$

as against π for the circle of equal diameter two units.

Considering the area of curves formed from odd-sided regular polygons whose equation is

$$p = a + \sum_{n=1}^{\infty} a_{nm} \cos mn\psi, \quad (n \text{ odd}),$$

the least area occurs for $m = 3$, giving the curve of Fig. 2. This area is $2(\pi - \sqrt{3})a^2$, which for $a = 1$ is less than $15\pi/16$. Hence the curve of Fig. 2 has the smallest area of curves discussed in this note. As $m \to \infty$ the curve tends to the circle $p = a$.

In conclusion, I would like to thank Mr. D. G. Kendall for his kind assistance and encouragement in the writing of this note.

M. J. Kearsley.

1704. Thus he [Arthur Conan Doyle] reached Paris with a book on conic sections in his hand, Edgar Allan Poe in his head, and twopence in his pocket. —John Dickson Carr, *The Life of Sir Arthur Conan Doyle* (1949), p. 32.

WHERE DOES THE CAR ENGINE STOP?

By J. C. Cooke.

Introduction.

Crawling under a very old six-cylinder car to find out why the self-starter kept jamming, I found that the large gear wheel round the outside of the fly-wheel was worn very badly in three regions spaced at 120° intervals round its outside. Presumably then the engine always stops in one of three positions, and every time the self-starter is used it crashes always into one of these three places. It is an interesting little problem to see just why the engine always stops in these places and to find out mathematically where they are.

Assumptions.

We make some assumptions : (1) The compression ratio is reasonably large, say 6 ; (2) We use the adiabatic law $pv^{\gamma} = \text{constant}$ to determine the pressure inside the cylinder. This assumes no heat transfer between the gas and the cylinder walls in the time between the start of the compression and the time the engine stops. γ is usually taken to lie between 1·3 and 1·4 ; (3) There is no leakage of gas past the piston and valves. If there is leakage it would roughly correspond to a reduction in compression ratio ; (4) The friction is neglected ; (5) The connecting rod is long compared to the stroke of the piston. This approximation does not introduce much error, and not to make it complicates the analysis.

The crankshaft turning moment.

If θ is the angle the crankshaft has turned through from top dead centre, $2a$ is the total stroke, A is the cross-section of the cylinder and V is the volume in the cylinder head above top dead centre, then the volume of gas enclosed in any one cylinder (assuming that both valves are closed) is $V + aA(1 - \cos\theta)$, assuming a very long crankshaft. The compression ratio is $(V + 2aA)/V$.

We have then that the pressure P on the piston is given by

$$P = \frac{C}{\{V + aA(1 - \cos\theta)\}^{\gamma}},$$

where the constant C is determined by the conditions at the moment compression starts.

If the crankshaft is long, then the turning moment on the crankshaft is $Pa \sin\theta$.

Now, the car has six cylinders and the cranks are set uniformly round the crankshaft at 120° intervals, not necessarily in the order that we come to them on going from cylinder to cylinder starting from number one.

The Complete Engine.

We draw a figure showing the positions of the cranks at any time, denoting by the letters S, C, I, E what "stroke" any particular piston is on. The letters stand for Suction, Compression, Ignition and Exhaust respectively. For $0 < \theta < 60°$, the situation will be as in Fig. 1, for $60° < \theta < 120°$, as in Fig. 2. For $\theta > 120°$ the figures repeat themselves. There are two cylinders represented by each radius in the figures each with its own letter.

Now, one or other of the valves is open for the strokes E and S, and both are shut for C and I, and so for our purpose we need only consider three of the cylinders (one from each pair) ; the others have open valves. (For the moment we assume that valves open and shut at top or bottom dead centre ; actually the exhaust and inlet valves usually open early and close late and we shall consider this later.)

Hence the total turning moment on the crankshaft is

$$\Sigma \frac{Ca \sin (\theta + 120^\circ . r)}{\{V + aA - aA \cos (\theta + 120^\circ . r)\}^\gamma},$$

where r takes the values 0, 1, 2.

Putting $(V + aA)/aA = y$ so that $y > 1$, we have for the turning moment M the equation

$$M = \Sigma \frac{\sin (\theta + 120^\circ . r)}{\{y - \cos (\theta + 120^\circ . r)\}^\gamma}, \quad r = 0, 1, 2, \quad \text{..................(1)}$$

dropping a constant positive multiplier.

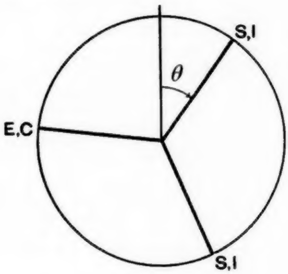

Fig. 1.

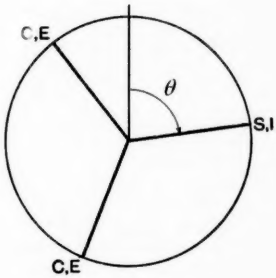

Fig. 2.

The Position of Equilibrium.

For equilibrium we must have $M = 0$, and for stability $dM/d\theta < 0$, since, if equilibrium is to be stable at the point where $M = 0$ a small increase in θ should produce a negative turning moment.

The solutions of the equation $M = 0$ depend on y and γ but there are two obvious ones in the range $0 \leqslant \theta < 120^\circ$, namely $\theta = 0$ and $\theta = 60^\circ$. These are symmetrical positions and we shall see that the position $\theta = 0$ is unstable and $\theta = 60^\circ$ is stable.

The Position $\theta = 0$.

We have

$$\frac{dM}{d\theta} = \Sigma \frac{y \cos (\theta + 120^\circ . r) - \{1 + (\gamma - 1) \sin^2 (\theta + 120^\circ . r)\}}{\{y - \cos (\theta + 120^\circ . r)\}^{\gamma + 1}}.$$

For $\theta = 0$ this gives, on putting $y = 1 + x$ so that $x > 0$,

$$\frac{dM}{d\theta} = \frac{1}{x^\gamma} - \frac{1 \cdot 5 + x + 3\gamma/2}{(1 \cdot 5 + x)^{\gamma + 1}}.$$

Hence, this position is unstable if

$$\left(\frac{1 \cdot 5 + x}{x}\right)^{\gamma + 1} > \frac{1 \cdot 5 + x + 3\gamma/2}{x}$$

or, on putting $1 \cdot 5 + x = xz$, so that $z > 1$, we must have

$$E \equiv z^{\gamma + 1} - z(\gamma + 1) + \gamma > 0.$$

This is certainly true if $z > 1$ because E is zero when $z = 1$ and $dE/dz > 0$ if $z > 1$.

Hence the position $\theta = 0$ is unstable.

The Position $\theta = 60°$.

The case $\theta = 60°$ requires further consideration of the figures. At this position, two pistons are at their lowest points, *i.e.* one has finished Suction and is starting Compression, and one has finished Ignition and is starting Exhaust. However, as the exhaust valve always opens early and the inlet valve always closes late, the angle often being as much as 45° in each case, we may say that both the cylinders whose pistons are at bottom dead centre have both their valves open, not only at the bottom but for a considerable angle either side of it. Consequently, in the neighbourhood of $\theta = 60°$ the second term of M in the equation (1) is no longer present. $\theta = 60°$ is however, still a root of this modified equation, *viz.*,

$$M = \Sigma \frac{\sin(\theta + 120° \,.\, r)}{\{y - \cos(\theta + 120° \,.\, r)\}^{\gamma}}, \quad r = 0, 2 \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

In this case, for $\theta = 60°$, we easily have

$$\frac{dM}{d\theta} = \frac{y - 2\{1 + 3(\gamma - 1)/4\}}{\{y - 1/2\}^{\gamma+1}}$$

and $dM/d\theta < 0$ if $y < (3\gamma + 1)/2$.

This condition is most stringent for the case $\gamma = 1$, and for this value gives $y < 2$ as the condition for stability. If this holds then the condition will certainly continue to hold for all permissible values of γ.

Now the compression ratio is $(V + 2aA)/V$ and $y = (V + aA)/aA$; this makes the value of the compression ratio equal to $(y+1)/(y-1)$. Hence, for stability the compression ratio must be greater than 3, a condition fulfilled in any motor car engine. Hence $\theta = 60°$ gives a stable position of rest.

Other Solutions.

The question arises as to whether $\theta = 0$, $\theta = 60°$ are the only positions of equilibrium for $0 \leqslant \theta < 120°$. I am satisfied that equation (1) has no other solutions but these for all $y > 1$ and for all $\gamma \geqslant 1$, but I am unable to prove it. So also equation (2) has a stable solution $\theta = 60°$ and, for the values of y and γ for ordinary cars, another unstable one near $\theta = 0$. I have been unable to prove these results, but I have plotted the curves of equations (1) and (2) for $y = 1{\cdot}2$, $1{\cdot}4$, and for $\gamma = 1{\cdot}3$ and $\gamma = 1{\cdot}4$. These curves show clearly that equation (1) has no other roots but $\theta = 0$, $\theta = 60°$, and equation (2) has another root between $\theta = 0$ and $\theta = 10°$ which, even if it were permissible to use equation (2) in this range of values of θ (which it is not in any normal car) would give an unstable position. It is obvious from the way the curves run that the same will apply for any value of γ between $1{\cdot}3$ and $1{\cdot}4$ and it can also be fairly reliably deduced that nothing in this respect is changed for all y between $1{\cdot}1$ and $1{\cdot}5$, *i.e.* for all compression ratios between 21 and 5.

Hence, we deduce that the engine stops at the position $\theta = 60°$, or two other similar positions $\theta = 180°$ and $300°$.

"*Comparison with experiment.*"

As far as I can judge (by looking at the position of the contact breaker cam) the car in question appears to stop always in the position where θ is about 50°. Friction may possibly have something to do with this, but I think it is more likely to be gas leakage. (Remember that the car is an old one.) Of the two cylinders concerned, the one going up is about to compress the gas further, but the one going down *has* compressed the gas. This latter therefore is more likely to have made some of the gas leak and consequently at $\theta = 60°$ the downward pressure on this one is not now as great as that on the other piston. From this we should expect θ to be somewhat less than its theoretical value of 60°.

J. C. C.

THE SOLUTION OF INFERENTIAL PROBLEMS BY BOOLE ALGEBRA.

By T. J. Fletcher.

It is nearly a hundred years since Boole published *An Introduction to the Laws of Thought* (**1**). An attempt to reduce the art of logical deduction to algebra was a bold venture, and perhaps not all of Boole's original hopes have been fulfilled. Nevertheless, Boole algebra has become a well-established branch of mathematics which has given rise to theoretical masterpieces such as *Principia Mathematica* (**2**), and has found practical applications in electrical circuitry, switching problems, insurance law and statistics.

It is good that students should realise early that mathematics is not merely concerned with one unique algebra, but that as many algebras exist as the ingenious care to construct. Boole algebra is an algebra which can be learnt very quickly, and used to solve problems which are more difficult by other methods. The examples in the classical works (**1, 3, 4**) mostly have a rather old-fashioned air these days; but a type of inferential puzzle is in vogue at the moment which provides excellent problems, of a light-hearted kind, to solve by symbolic methods. This paper summarises the algebra that is necessary and gives a few specimen solutions.

The algebra of classes.

We commence by giving an outline of the algebra required. We shall eventually use an algebra of propositions; but the ideas are more easily understood if the algebra is introduced as an algebra of classes. A letter is used to denote the class of individuals possessing a certain property. The product notation xy is used to denote the class consisting of those individuals which are members of both x and y. The notation $x+y$ is used to denote the class whose members are either members of x or members of y (or of both).

It is easy to see that the following algebraic relations are true:

$$x+y=y+x, \qquad xy=yx; \qquad \text{.......................(1)}$$

$$(x+y)+z=x+(y+z), \quad (xy)z=x(yz); \qquad \text{.....................(2)}$$

$$x(y+z)=xy+xz. \qquad \text{....................................(3)}$$

It is possible to define subtraction and division in this algebra, but it is not worth while. In so far as addition and multiplication are concerned the familiar laws of normal algebra can all be deduced from equations (1), (2), (3), and these laws are also valid in the new algebra we are constructing.

Now $x^2=xx$, which denotes exactly the same class as the symbol x, hence

$$x^2=x, \text{ and generally } x^n=x. \qquad \text{..........................(4)}$$

Similarly, $2x=x+x$, which also denotes the same class as x, hence

$$2x=x+x=x, \text{ and generally, } x+x+\ldots+x=x. \qquad \text{................(5)}$$

Equations (4) and (5) are called the *laws of absorption*.

The next relation is not so obvious:

$$x+yz=(x+y)(x+z). \qquad \text{................................(6)}$$

Multiplying out the right-hand side we get $x^2+xz+yx+yz$. The first term is x, by equation (4), and the terms xz and yx are both included already in the class x (as they merely denote those members of the class x possessing some other property in addition). Hence the right-hand side of equation (6) reduces to the left-hand side.

If we regard multiplication and addition as dual operations, it can be seen

that the algebra which we are developing is dual, in a way that ordinary algebra is not. The members of equation (1) are duals, and so are the members of (2); (4) and (5) are dual, and (6) is the dual of (3). Ordinary algebra is not dual in this way because equation (6) is false in it.

Symbols are needed to denote the *null-class* (that is, the class with no members, or the class whose members do not exist) and the *universal-class* (that is, the class containing all the individuals under discussion). These are denoted by 0 and 1 respectively. These symbols are not, of course, the 0 and 1 of ordinary arithmetic, but they obey much the same formal laws.

The class consisting of those individuals which are not members of x is denoted by x', and the following results may be verified:

$$xx'=0, \qquad x+x'=1; \quad \text{......(7)}$$

$$(x')'=x; \quad \text{......(8)}$$

$$0\,.\,x=0, \qquad 1+x=1; \quad \text{......(9)}$$

$$1\,.\,x=x, \qquad 0+x=x; \quad \text{......(10)}$$

$$1'=0, \qquad 0'=1. \quad \text{......(11)}$$

When dualising by interchange of the operations of multiplication and addition the symbols 0 and 1 must be interchanged at the same time. Observe that $xy=1$ implies both $x=1$ and $y=1$; while $x+y=0$ implies both $x=0$ and $y=0$.

The list of important results is completed by de Morgan's laws,

$$(xy)'=x'+y', \qquad (x+y)'=x'y'. \quad \text{......(12)}$$

The significance of these relations is made most clear by drawing circle diagrams. In such diagrams the members of a class x are represented by the points inside a circle, and the members of x' are represented by the points outside.

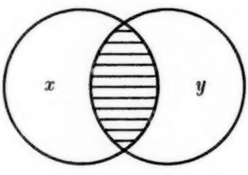

FIG. 1.

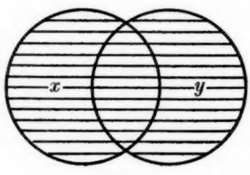

FIG. 2.

In Fig. 1, the class xy is shaded. The remaining region of the plane represents $(xy)'$ and is identical with the region included in x' or y' or both. This illustrates the first of equations (12). In Fig. 2, the class $x+y$ is shaded. The remaining region of the plane represents $(x+y)'$ and is identical with the region outside both x and y. This illustrates the second of equations (12).

The algebra of propositions.

The algebra of propositions is exactly the same formally as the algebra of classes just described; but a different interpretation is placed upon the symbols.

A letter is used to denote a proposition. For example, p may denote "Trafalgar Square is in London", and q may denote "Pigs have wings". pq denotes the two propositions p and q taken in *conjunction*; that is, it denotes the proposition asserting p and q simultaneously. In the example given, pq would be false, because p is true but q is false. $p+q$ denotes the two propositions p and q taken in *disjunction*; that is, it denotes the pro-

position asserting either p or q. In the given example $p+q$ is true, because whilst q is false, p is true, and $p+q$ asserts *either p or q*.

It is easy to see that equations (1)–(6) of the previous section hold with this new interpretation of the symbols.

A proposition which is false is said to have the *truth-value* 0, and $p=0$ means " p is false ". A proposition which is true is said to have the *truth-value* 1, and $p=1$ means " p is true ". p' denotes the negation of p. If p is true, p' is false, and *vice versa*.

Equations (7)–(12) of the previous section can now be interpreted in terms of propositions. For example, the two equations (7) mean respectively, " A statement x cannot be both true and false at the same time " and " a statement x is either true or false ". The two equations (9) mean respectively, " The conjunction of a false statement with any statement is false " and " The disjunction of a true statement with any statement is true ". The laws of de Morgan now express the laws of negation of compound propositions ; and a little consideration will convince the reader that they are merely an expression of common usage in symbolic form.

In the algebra of propositions each symbol has the value 0 or 1, and for this reason it can be described as a *two-valued algebra*. This is not the case with the algebra of classes. The task in a problem in the algebra of propositions is generally to determine the truth-values of the unknowns. The algebra as described so far is sufficient to solve a wide range of problems. It will not, of course, do anything that could not be done in some other way. Trial and error, or systematic enumeration of cases will solve most puzzles ; but the algebra enables them to be approached with a standard technique, and has the advantage that it will reveal any alternative answers which a solution by trial and error tends to pass over. It is also a most convenient weapon for the composer of puzzles ; the puzzle can be worked out entirely in symbols and be given a verbal cloak afterwards.

Example 1 (Hubert Phillips, 5).

Out of six boys, two were known to have been stealing apples. But who? Harry said, " Charlie and George ". James said, " Donald and Tom ". Donald said, " Tom and Charlie ". George said, " Harry and Charlie ". Charlie said, " Donald and James ". Tom couldn't be found.

Four of the boys interrogated named one miscreant correctly. The fifth had lied outright.

Who stole the apples?

Let H, J, D, G, C, T denote the propositions " Harry, James, Donald, George, Charlie, Tom did it " respectively. As each of the five statements made is false when taken in conjunction,

$$CG=DT=TC=HC=DJ=0. \qquad \text{(1)}$$

Four of the five statements are true taken in disjunction, but the fifth is false, because the fifth boy had lied outright. Thus

$$(C+G)(D+T)(T+C)(H+C)(D+J)=0. \qquad \text{(2)}$$

Multiplying out equation (2) and using (1),

$$CD=0. \qquad \text{(3)}$$

But one set of four out of the five statements in disjunction are all true. Hence

$$\sum_{5}(\quad)(\quad)(\quad)(\quad)=1, \qquad \text{(4)}$$

N

where the empty brackets stand for a selection of four of the brackets on the left-hand side of equation (2) and the five possible combinations are summed. Multiplying out equation (4) and using equations (1) and (3) this readily reduces to

$$CJ = 1.$$

This means that Charlie and James stole the apples.

Example 2.

I possess three counters A, B, C, which are coloured red, white and blue, but not necessarily respectively. Of the following statements about the counters one only is true ;

A is red : B is not red : C is not blue.

What colour is each counter?

In problems of this sort it is useful to employ a suffix notation ; A_r will denote the proposition " A is red ", and similarly for the others. The given statements are then A_r, B_r' and C_b'. From the nature of the problem we at once have many relations, such as

$$A_r + B_r + C_r = 1, \quad \text{.....(1)}$$

$$A_r + A_b + A_w = 1, \quad \text{.....(2)}$$

$$A_r B_r \quad = 0, \quad \text{.....(3)}$$

$$A_r A_b \quad = 0. \quad \text{.....(4)}$$

The first expresses the fact that one of the three counters is red, the second that counter A is one of the three colours, the third that A and B cannot both be red, and the fourth that A cannot be both red and blue. It is convenient to call these the *exclusion relations*, and we shall not state them explicitly in future problems unless there is particular reason to do so. In equation (3) it is possible to regard r as a dummy suffix to be summed over the range r, w, b ; and in equation (4) a similar convention can be applied to the repeated letter A. The four equations bear some resemblance to the equations between the direction cosines of a set of orthogonal axes. Only a few of these relations are needed for any one problem, but they are all mentioned here for sake of completeness.

The working of this particular problem is as follows. Since one and only one of the three given statements is true, we know one of three alternatives to be the case : the symbolic expression of this is

$$A_r B_r C_b + A_r' B_r C_b' + A_r' B_r' C_b = 1. \quad \text{.....(5)}$$

The first term of this vanishes by equation (3). When we multiply equations (1) and (5) together the only term which does not vanish is

$$A_r' B_r C_b' = 1. \quad \text{.....(6)}$$

This gives the truth values of the original statements, and we see successively that B is red, C is white and A is blue. These conclusions follow easily enough on inspection of equation (6) ; but to see them emerging by purely algebraical methods, multiply the two sides of (6) by the two sides of the following equations :

$$A_w + B_w + C_w = 1,$$

$$A_b + B_b + C_b = 1.$$

This gives only one term which does not vanish,

$$A_r' B_r C_b' A_b C_w = 1,$$

and for a product to be 1 each factor must be 1.

Example 3 (Hubert Phillips, 6).

Alice, Brenda, Cissie and Doreen competed for a scholarship.
" *What luck have you had?* " *someone asked them.*
Said Alice ; " *Cissie was top. Brenda was second.*"
Said Brenda ; " *No : Cissie was second and Doreen was third.*"
Said Cissie ; " *Doreen was bottom. Alice was second.*"
Each of the three girls had made two assertions, of which only one was true.
Who won the scholarship?

Employing the obvious suffix notation, in which, for example, A_1 denotes " Alice was first ", the statements of the girls together with the information we have about their statements yield the following :

from Alice, $C_1B_2=0$ and $C_1+B_2=1$, giving $C_1B_2'+C_1'B_2=1$;
from Brenda, $C_2D_3=0$ and $C_2+D_3=1$, giving $C_2D_3'+C_2'D_3=1$;
from Cissie, $D_4A_2=0$ and $D_4+A_2=1$, giving $D_4A_2'+D_4'A_2=1$.

Multiplying these together,

$$(C_1B_2'+C_1'B_2)(C_2D_3'+C_2'D_3)(D_4A_2'+D_4'A_2)=1$$

and expanding and using the exclusion relations, only one term is left on the left-hand side,

$$C_1B_2'C_2'D_3D_4'A_2=1.$$

This means Cissie was first, Alice second, Doreen third and Brenda fourth.

A further difficulty arises in problems of the sort in which the notion of " namesake " is involved. For example, we may have a question about Messrs. Flute, Harp, Trombone and Sackbut, who play the flute, harp, trombone and sackbut. The difficulty arises when we are told that " the namesake of Mr. Flute's instrument plays the sackbut ". Denoting " Mr. Flute plays the trombone " by the notation F_t, etc., we have the information that $F_xX_s=1$ for some x. That is to say, there is some instrument x for which it is true that " Mr. Flute plays the x and Mr. X plays the sackbut ". The technique to employ in such problems is to introduce unknowns as they are needed, and sum over the total possible range of letters ; in this case four. Three of the four terms F_xX_s are zero, and the remaining one has the value 1. In practice many terms can be neglected at sight because they vanish by exclusion relations. An example should make the idea clear.

Example 4 (Hubert Phillips, 5).

Four members of my Club—Messrs. Albert, Charles, Frederick and Dick—have recently been knighted, so now their friends have had to learn their Christian names.

These are a little worrying : for it transpires that the surname of each of the four new knights is the Christian name of one of the others.

Dick is not the Christian name of the member whose surname is Albert. The Christian name of the member whose surname is Frederick is the surname of the member whose Christian name is the surname of the member whose Christian name is Charles.

What is the Christian name of the member whose surname is Dick?

Using capital letters A, C, F and D for the Christian names, and small suffixes for the surnames, the information is

$A_a=0$, etc., for all letters, ...(1)

$D_a=0$, ...(2)

and there is an X and a Y such that

$$X_f Y_x C_y = 1. \qquad \ldots\ldots\ldots\ldots (3)$$

We sum equation (3) for all X and Y and drop the terms which vanish by (1) and (2).

$$\begin{aligned} 1 &= \sum_{x,y} X_f Y_x C_y \\ &= \sum_x X_f (A_x C_a + F_x C_f + D_x C_d) \\ &= D_f A_d C_a. \end{aligned}$$

Therefore Mr. Dick's Christian name is Albert, and the calculation also reveals the other names as well.

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T.J.F.

1705. She [Florence Nightingale] had written to Clarkey that mathematics gave her certainty : mathematics required hard work, and perhaps she would find life more satisfactory, be more satisfactory herself, if she studied mathematics. She confided in Aunt Mai and they began to work together, getting up before it was light to avoid disturbing the routine of the house. She became wildly happy. If only her parents would let her have lessons, if only they could be persuaded to let her study mathematics instead of doing worsted work and practising quadrilles.

In March 1840 Aunt Mai wrote Fanny [Florence's mother] a cautious letter. .... I write to ask you if you in any way object to a mathematical master, if one can find a clean middle aged respectable person. . . .

Fanny raised difficulties about the master. . . . Aunt Mai persevered, soothing, reassuring, producing a married man, a clergyman, one accustomed to teaching young ladies, a chaperone. [Her mother would not agree, but for a short time Florence received lessons in her uncle's library.]

She found (about 1860) statistics " more enlivening than a novel " and loved to bite on a hard fact " . . . however exhausted Florence might be the sight of long columns of figures was " perfectly reviving " to her.—Cecil Woodham Smith, *Florence Nightingale* 1820–1910. [Per Prof. H. T. H. Piaggio.]

1706. The Examples are, for the most part, reprinted *verbatim* from the papers in which they were set ; in a few the language has been altered, or the theorem involved has been generalized ; several, however, have defied all attempts at improvement, and now stand in their unintelligibility as a warning, to the Candidate for Mathematical Honors, of the ordeal he may have to pass through.—Tait and Steele, *Dynamics of a Particle*, preface to 2nd edition (1865), pp. viii-ix.

ON THE LIMIT OF THE RATIO OF SIN X TO X.

BY R. L. GOODSTEIN.

THE fallacies in the traditional approach to the problem of evaluating

$$\lim_{x\to 0} \frac{\sin x}{x}$$

in a pre-calculus course have been well brought out in an interesting article by Mr. Turton * where the limit is obtained by means of polynomial approximations to the area of a circle. On re-reading the article recently it occurred to me that, at very little extra cost in effort and depth of ideas, a rigorous evaluation of the limit was possible which was completely free from appeals to the physico-spatial properties of geometrical figures and assumptions of symmetry and continuity which Mr. Turton makes.

We take for granted only the right-angle triangle properties of the trigonometric functions including the addition formulae for (small enough) angles. Angles are measured by rational or irrational submultiples of a right angle, and, for brevity (and another reason that will become apparent later) we denote the sine, cosine and tangent of an angle of α degrees by $S(\alpha)$, $C(\alpha)$ and $T(\alpha)$ respectively.

1. We start by proving a number of subsidiary results.

1.1. For $0 \leqslant \alpha < \beta \leqslant 90$,

$$C(\alpha) - C(\beta) = 2S(\tfrac{1}{2}(\beta - \alpha)) \,.\, S(\tfrac{1}{2}(\alpha + \beta)) > 0$$

and

$$S(\beta) - S(\alpha) = 2S(\tfrac{1}{2}(\beta - \alpha)) \,.\, C(\tfrac{1}{2}(\alpha + \beta)) > 0,$$

so that $S(\alpha)$ is steadily increasing and $C(\alpha)$ steadily decreasing, for $0 \leqslant \alpha \leqslant 90$.

1.2. From $S^2(\alpha) + C^2(\alpha) = 1$, it follows that $|C(\alpha)| \leqslant 1$, $|S(\alpha)| \leqslant 1$ and from

$$C^2(\alpha) - S^2(\alpha) = C(2\alpha),$$

we have $S^2(\alpha) < C^2(\alpha)$ for $0 < \alpha < 45$, so that $T(\alpha) < 1$. Further, from

$$C(\alpha) = C^2(\tfrac{1}{2}\alpha) - S^2(\tfrac{1}{2}\alpha) < C^2(\tfrac{1}{2}\alpha),$$

it follows that, if $0 < \alpha \leqslant 1$, then

$$C(1) \leqslant C(\alpha) < \{C(\alpha/2^n)\}^{2^n},$$

and so

$$1 > C(\alpha/2^n) > \{C(1)\}^{1/2^n}.$$

Write $a = 1/C(1)$; given any positive d, for all sufficiently great n,

$$1 < a < 1 + nd < (1 + d)^n$$

and so

$$1 < a^{1/n} > 1 + d,$$

which proves that $a^{1/n} \to 1$ and so $\{C(1)\}^{1/2^n} \to 1$.

1.3. It follows that $C(\alpha/2^n) \to 1$ *uniformly* in α for $0 < \alpha \leqslant 1$, and so for $0 < \alpha \leqslant A$ for any positive A, since $A/2^k \leqslant 1$ for all sufficiently great k; hence too $S(\alpha/2^n) \to 0$ uniformly in α.

1.4. Since $C(x)$ is steadily decreasing for $0 \leqslant x \leqslant 90$, if $0 < x < 1/2^n$

$$1 > C(x) > C(1/2^n),$$

* The December *Gazette* for 1946, pp. 282–286.

which proves that $C(x)\to 1$ when $x\to 0$. Similarly, $S(x)\to 0$ when $x\to 0$. The same argument shows that

$$C(\alpha/n)\to 1$$

uniformly in α and

$$S(\alpha/n)\to 0$$

uniformly in α.

1.5. $S(\alpha)$ and $C(\alpha)$ are continuous functions, for

$$\begin{aligned}|S(\beta)-S(\alpha)| &= 2S(\tfrac{1}{2}|\alpha-\beta|)\,.\,|C(\tfrac{1}{2}(\alpha+\beta))| \\ &\leqslant 2S(\tfrac{1}{2}|\alpha-\beta|) \\ &\to 0,\end{aligned}$$

when $\beta-\alpha\to 0$, *i.e.* $\beta\to\alpha$, and similarly $C(\beta)-C(\alpha)\to 0$ when $\beta\to\alpha$.

2. If we join the end points of a circular arc AB by an inscribed polygon $AA_1A_2\ldots A_nB$, we call the sum of the sides of the polygon a *lower sum* for the arc, or specifically the lower sum on the subdivision A_r, $1\leqslant r\leqslant n$. If the tangents to the arc at $A, A_1, A_2, \ldots A_n, B$ meet in turn at $T_1, T_2, \ldots, T_n$, then the sum of the sides of the circumscribed polygon $AT_1T_2\ldots T_nB$ is called an *upper sum* for the arc, or specifically the upper sum on the subdivision A_r, $1\leqslant r\leqslant n$.

2.1. Increasing the number of points of subdivision increases a lower sum and decreases an upper sum. For if we insert a point of subdivision N between the points L and M of a unit circular arc, where LN, MN subtend angles 2α, 2β at the centre of the arc ($0<\alpha<45$, $0<\beta<45$), then the extra point N *increases* the lower sum by

$$2\{S(\alpha)+S(\beta)-S(\alpha+\beta)\}=2S(\alpha)(1-C(\beta))+2S(\beta)(1-C(\alpha))>0$$

and decreases the upper sum by

$$2\{T(\alpha+\beta)-T(\alpha)-T(\beta)\}=2\{T(\alpha)+T(\beta)\}T(\alpha)T(\beta)/\{1-T(\alpha)T(\beta)\}>0.$$

3. O is the centre of a circular arc AB and A_r, $1\leqslant r\leqslant n-1$ is a subdivision of AB such that the radii OA_r, $1\leqslant r\leqslant n-1$, divide the angle AOB into n equal parts; s_n, t_n are the lower and upper sums on the subdivision A_r. We shall prove that s_n and t_n tend to a common limit which we call the *length* of the arc AB.

3.1. If the angle AOB contains α degrees and $AO=1$, then $s_n=2nS(\alpha/2n)$, $t_n=2nT(\alpha/2n)$ and so

$$s_n/t_n=C(\alpha/2n)<1$$

and $s_n/t_n\to 1$ uniformly in α.

By 2.1, for any m, $n\geqslant 1$,

$$s_m\leqslant s_{mn}<t_{mn}\leqslant t_n,$$

i.e.

$$s_m<t_n.$$

Hence, for $0\leqslant\alpha\leqslant A<90$,

$$0<\frac{t_n-s_n}{2T(\frac{1}{2}A)}\leqslant\frac{t_n-s_n}{t_1}<\frac{t_n-s_n}{s_n}=\frac{t_n}{s_n}-1,$$

which proves that $t_n-s_n\to 0$ uniformly in α; but

$$-(t_m-s_m)<s_m-s_n<t_n-s_n,$$

so that s_n converges uniformly in α, to $\lambda(\alpha)$, say, and since $t_n-s_n\to 0$, t_n also converges to $\lambda(\alpha)$.

3.2. We prove next that $\lambda(\alpha)$ is proportional to α.

Let $k=\lim nS(1/n)$, the limit existing by 3.1.

If $\alpha = p/q$ where p, q are positive integers, then

$$\begin{aligned}\lambda(p/q) &= \lim 2nS(p/2qn)\\ &= \lim 2pnS(1/2qn)\\ &= (p/q)\,.\,\lim 2qnS(1/2qn) = (p/q)k\,;\end{aligned}$$

thus for rational values of α,

$$\lambda(p/q) = (p/q)\,k.$$

Since $\lambda(\alpha)$ is the limit of a *uniformly* convergent sequence of continuous functions, therefore $\lambda(\alpha)$ itself is a continuous function, and so if α is the irrational limit of a convergent sequence of rationals α_n,

$$\lambda(\alpha) = \lim \lambda(\alpha_n) = \lim k\alpha_n = k\alpha,$$

which proves that $\lambda(\alpha) = \alpha$ for all values of α.

Thus the length of the arc of a unit circle subtending an angle of α degrees at the centre is $k\alpha$, where $k = \lim nS(1/n)$.

3.3. $$\lim_{\theta\to 0} \{S(\theta)/\theta\} = k.$$

For any positive θ, however small, we can choose n so that

$$1/(n+1) < \theta \leqslant 1/n$$

and so

$$S(1/(n+1)) < S(\theta) \leqslant S(1/n)$$

whence

$$\frac{(n+1)S(1/(n+1))}{1+1/n} < \frac{S(\theta)}{\theta} < \left(1+\frac{1}{n}\right) nS(1/n)$$

and therefore $S(\theta)/\theta \to k$ when $\theta \to 0$.

4. We define the function $\sin x$ to be half the chord of a unit circle which cuts off an arc of length $2x$, and the function $\cos x$ to be the distance of this chord from the centre of the circle; $\tan x$ may be taken to be $\sin x/\cos x$ or may be defined directly in terms of the tangents at the ends of the arc. Since an arc of length $2x$ subtends an angle of x/k degrees at the centre, it follows that $\sin x = S(x/k)$, $\cos x = C(x/k)$ and $\tan x = T(x/k)$.

These definitions are appropriate for values of x between 0 and $90k$; to extend the range of definition we may either *postulate* the addition formulae

$$\sin(x+y) = \sin x \cos y + \cos x \sin y, \quad \cos(x+y) = \cos x \cos y - \sin x \sin y$$

for *all* values of x and y, or define $\sin x$ and $\cos x$ to be periodic with period $360k$ and cover the range from $90k$ to $360k$ by the explicit definitions

$$\begin{aligned}\sin(180k - x) &= \sin x & \cos(180k - x) &= -\cos x\\ \sin(180k + x) &= -\sin x & \cos(180k + x) &= -\cos x.\end{aligned}$$

Writing $y = x/k$, we have

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{y\to 0}\frac{S(y)}{ky} = \frac{1}{k}\lim_{y\to 0}\frac{S(y)}{y} = 1,$$

i.e.

$$\lim_{x\to 0}\frac{\sin x}{x} = 1.$$

4.1. We observe in passing that the foregoing definition of $\sin x$ makes no reference to radians. It is of the utmost importance in the development of the calculus to emphasise the fact that the relation $y = \sin x$ is a relation between two *numbers* x and y and not a relationship restricted to an angle measurement, so that for instance arc sin 1 is a pure number, not so many radians. There is, of course, no objection to the introduction of a second unit for the

measurement of angles; and if we define a radian to be an angle of $1/k$ degrees it follows that, in a unit circle, an arc of length x subtends an angle of x radians at the centre of the circle, and that $\sin x$ is equal to the sine of an angle of x radians, but radians are just as irrelevant to the calculus as are degrees.

4.2. If we denote the length of a semicircle of unit radius as usual by π, it follows from 3.2 that the value of π is given by

$$\pi = 180k,$$

where $k = \lim nS(1/n)$.

4.3. Since $s_n \to k\alpha$ and $t_m > s_n$, it follows that, for any m, $t_m \geqslant k\alpha$ and so $t_1 > t_2 \geqslant k\alpha$, *i.e.* $t_1 > k\alpha$. Similarly, $s_1 < k\alpha$. Thus

$$2S(\tfrac{1}{2}\alpha) < k\alpha < 2T(\tfrac{1}{2}\alpha) \quad \text{for } 0 < \alpha < 180,$$

and taking $\alpha = 2x/k$, we have

$$\sin x < x < \tan x \quad \text{for } 0 < x < \tfrac{1}{2}\pi.$$

5. The length of the arc of a circle of radius r which subtends an angle of x radians at its centre is by definition

$$\lim 2rn \sin(x/2n) = rx.$$

To determine the area bounded by the arc and the radii through its extremities we consider inscribed and circumscribed polygons which together with the radii contain areas

$$r^2 n \sin(x/2n) \quad \text{and} \quad r^2 n \tan(x/2n)$$

respectively, and these areas have the common limit $\frac{1}{2}r^2 x$.

6. We conclude by showing that if σ is the sum of the sides of any polygon P (not necessarily regular) inscribed in AB, then σ converges to the length of the arc AB provided that the length of the greatest side of P tends to zero.

Let the arc AB subtend an angle of α radians at the centre of a unit circle, and let σ_m be the lower sum for the arc on a subdivision

$$B_r, \quad 0 \leqslant r \leqslant p+1, \quad (B_0 = A,\ B_{p+1} = B),$$

such that the angle subtended at the centre by the arc $B_r B_{r+1}$, $0 \leqslant r \leqslant p$, is less than α/m radians. We have to prove that $\sigma_m \to 0$. Since s_n converges to α the length of the arc AB, given any positive integer N, we can choose n so that

$$\alpha - s_n < 1/N.$$

Let S be the lower sum on the subdivision formed by taking all the points B_r and all the points A_r of the subdivision on which s_n is formed. Then $\sigma_m \leqslant S$ and $s_n \leqslant S$ so that

$$\alpha - S < 1/N.$$

If $m \geqslant n$, at most one point A_s can fall between any two B_r, B_{r+1}, since the arc $B_r B_{r+1}$ subtends at the centre an angle less than that subtended by any of the arcs $A_r A_{r+1}$. If A_s falls between B_r and B_{r+1}, each of the chords $B_r A_s$, $A_s B_{r+1}$ and $B_r B_{r+1}$ is less than the arc $B_r B_{r+1}$ and so less than α/m, and therefore the change in a lower sum made by the introduction of the single point A_s is certainly less than $3\alpha/m$. Since there are $n-1$ points A_s, at most $n-1$ of the arcs $B_r B_{r+1}$ are broken by inserting the points A_s and therefore

$$S - \sigma_m < (n-1)3\alpha/m;$$

Hence, for $m \geqslant Nn$, $S - \sigma_m < 3/N$ and therefore $\alpha - \sigma_m < 4/N$ which proves that $\sigma_m \to \alpha$.

R. L. Goodstein.

ON THE SOLUTION OF LINEAR DIFFERENCE EQUATIONS.

By D. F. Lawden.

1. *Introduction.*

The solution of linear differential equations under given boundary and initial conditions has been greatly expedited in recent years by the use of the method of integral transforms. It is the purpose of this note to show that an entirely parallel method may be developed for the solution of linear difference equations and that this leads to a new view of the generating function of a sequence as a transform of the sequence, analogous to the Laplace transform of a function.

2. *Transform of a Sequence.*

Let $\{a_n\}$ be a sequence defined for $n=1, 2, \ldots$. Then, provided the power series

$$a(z)=a_1z^{-1}+a_2z^{-2}+\ldots, \qquad \text{(i)}$$

is convergent for $|z|>R$, $a(z)$ is called the "transform" of the sequence.

This series is the Laurent series for the function $a(z)$, which is regular and single valued in a neighbourhood of the point at infinity. Accordingly

$$a_n=\frac{1}{2\pi i}\int z^{n-1}a(z)\,dz, \qquad \text{(ii)}$$

the integral being taken around a contour enclosing the point at infinity. This is the "inversion theorem" for the transformation.

The representation of a function $a(z)$ by the power series (i) is unique and it follows that if an inverse transform of $a(z)$ can be found by any means, it is the only such inverse transform. In view of this (1, 1) relationship between sequences and their transforms, rather than use the inversion formula, we may draw up a table of common sequences and their transforms and use this to invert a given transform. A short table is given below:

a_n	$a(z)$
a^n	$a/(z-a)$
1	$1/(z-1)$
$n-1$	$1/(z-1)^2$
$1/n$	$\log\{z/(z-1)\}$
$1/n!$	$e^{1/z}-1$

Adopting a well-known notation, we write

$$a_n \supset a(z),$$

and then the following results are evident from the definition,

$$\alpha a_n \supset \alpha a(z), \quad (\alpha \text{ independent of } n),$$

$$a_n+b_n \supset a(z)+b(z),$$

$$a_{n+p} \supset z^p a(z)-a_1z^{p-1}-a_2z^{p-2}-\ldots-a_p.$$

3. *Method.*

Consider now the solution of a linear difference equation with constant coefficients, viz.:

$$a_0u_{n+p}+a_1u_{n+p-1}+\ldots+a_pu_n=v_n, \quad (n=1, 2, \ldots) \qquad \text{(iii)}$$

under the initial conditions $u_1=\alpha_1$, $u_2=\alpha_2$, $\ldots$ $u_p=\alpha_p$.

Taking the transform of both sides we obtain

$$a_0(z^p u - \alpha_1 z^{p-1} - \alpha_2 z^{p-2} - \dots - \alpha_p)$$
$$+ a_1(z^{p-1}u - \alpha_1 z^{p-2} - \alpha_2 z^{p-3} - \dots - \alpha_{p-1})$$
$$+ \dots + a_{p-1}(zu - \alpha_1) + a_p u = v,$$

$u(z)$ and $v(z)$ being the transforms of u_n and v_n respectively.

Solving for u, we have

$$(a_0 z^p + a_1 z^{p-1} + \dots + a_p)u = v + a_0(\alpha_1 z^{p-1} + \alpha_2 z^{p-2} + \dots + \alpha_p) + \dots + a_{p-1}\alpha_1.$$

The steps by which this equation has been obtained may be omitted once the simple rules for its formation have been learnt. These are strictly analogous to the rules for writing down the Laplace transform of the corresponding differential equation.

We can now solve for $u(z)$, and if the result is a rational function, break this up into partial fractions of the form $A(z-\alpha)^{-r}$, and then transform back to u_n by reference to a table of transforms.

This method may also be applied to the solution of a set of simultaneous linear difference equations, or to a difference equation involving two or more independent integral variables.

4. *Examples.*

(*a*) Solve

$$u_{n+3} - 6u_{n+2} + 11u_{n+1} - 6u_n = 1 + 4^n,$$

under the initial conditions $u_1 = 1$, $u_2 = 2$, $u_3 = 3$.

The transformed equation is

$$(z^3 - 6z^2 + 11z - 6)u = \frac{1}{z-1} + \frac{4}{z-4} + z^2 - 4z + 2\ ;$$

from which we calculate that

$$u(z) = \frac{1}{2}\cdot\frac{1}{(z-1)^2} - \frac{5}{12}\cdot\frac{1}{z-1} + \frac{3}{2}\cdot\frac{2}{z-2} - \frac{3}{4}\cdot\frac{3}{z-3} + \frac{1}{6}\cdot\frac{4}{z-4}.$$

Inverting, we have

$$u_n = \tfrac{1}{2}(n-1) - \tfrac{5}{12} + \tfrac{3}{2}\,.\,2^n - \tfrac{3}{4}\,.\,3^n + \tfrac{1}{6}\,.\,4^n$$

or
$$u_n = \tfrac{1}{2}n - \tfrac{11}{12} + 3\,.\,2^{n-1} - \tfrac{1}{4}\,.\,3^{n+1} + \tfrac{1}{6}\,.\,4^n.$$

(*b*) It is frequently the case in series that the terms alternate in sign, and this is allowed for in an expression for the general nth term by introduction of a factor $(-1)^n$. Consider a series in which the first r terms are all positive, the next r all negative, the next r all positive, and so on. Let u_n be the factor we must introduce into the nth term to allow for such fluctuations of sign. The sequence $\{u_n\}$ is

$$1, 1, \dots 1, -1, -1, \dots -1, 1, 1, \dots$$

and satisfies the difference equation

$$u_{n+2r-1} + u_{n+2r-2} + \dots + u_n = 0,$$

the initial conditions being $u_1 = u_2 = \dots = u_r = -u_{r+1} = \dots = -u_{2r-1} = 1$.

The transformed equation is

$$(z^{2r-1} + z^{2r-2} + \dots + 1)\,u(z) = z^{2r-2} + 2z^{2r-3} + \dots + (r-1)z^r + rz^{r-1}$$
$$+ (r-1)z^{r-2} + \dots + 3z^2 + 2z + 1$$
$$= \frac{z^{2r} - 2z^r + 1}{(z-1)^2}.$$

Hence

$$u(z)=\frac{z^{2r}-2z^r+1}{(z-1)(z^{2r}-1)}.$$

This is the generating function of the sequence, so that if it is expanded in a series of inverse powers of z, the coefficients are all unity and the signs are positive or negative in groups of r.

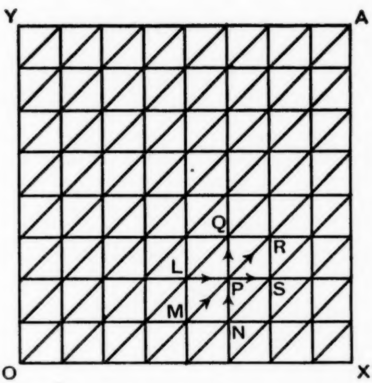
If ω_{2r} is the $2r$th root of unity with amplitude π/r, we may resolve $u(z)$ into partial fractions, thus

$$u(z)=-\frac{2}{r}\sum_{m=1}^{r}\frac{\omega_{2r}^{2m-1}}{(z-\omega_{2r}^{2m-1})(1-\omega_{2r}^{2m-1})},$$

and then inverting

$$u_n=-\frac{2}{r}\sum_{m=1}^{r}\frac{(\omega_{2r}^{2m-1})^n}{1-\omega_{2r}^{2m-1}}=\frac{2}{r}\sum_{m=1}^{r}\frac{\omega_{2r}^{(2m-1)n}}{\omega_{2r}^{2m-1}-1}.$$

(*c*) A square of side n units is subdivided into n^2 unit squares by parallels to the sides. The diagonal OA is drawn and also the diagonals which are parallel to OA of all the unit squares. We proceed to calculate the number of paths by which we may travel from O to A keeping to the lines of the figure, it being understood that at a junction such as P, a path may proceed in one of the directions PQ, PR, or PS only.



Take axes OX, OY as shown, and let P be the point (x, y). Let $u_{x,y}$ be the number of different paths connecting O and P. These paths arrive from three directions as indicated and it follows that the number of paths to P is the sum of the numbers of paths joining O to L, O to M and O to N. We have the relationship

$$u_{x,y}=u_{x,y-1}+u_{x-1,y}+u_{x-1,y-1}. \qquad \text{(iv)}$$

This is analogous to a partial differential equation, there being two independent variables x and y. If either x or y is negative it is convenient to define $u_{x,y}$ to be zero. This remark corresponds to the boundary conditions, since it defines $u_{x,y}$ over the boundaries $x=-1$, $y=-1$. The initial condition is $u_{0\,0}=1$, *i.e.* we suppose one path to enter at O. Equation (iv) is then universally true unless $x=y=0$.

Multiply (iv) throughout by p^{-y} and sum with respect to y between the limits 0 and ∞. We have

$$v_x = p^{-1}v_x + v_{x-1} + p^{-1}x_{x-1}, \quad (x \neq 0)$$

or

$$v_x = \frac{p+1}{p-1} v_{x-1}, \qquad \text{.........................(v)}$$

where

$$v_x(p) = \sum_{y=0}^{\infty} u_{xy} p^{-y}.$$

From (v) by repeated appeal

$$v_x = \left(\frac{p+1}{p-1}\right)^x v_0.$$

From (iv), putting $x=0$,

$$\begin{aligned} u_{0,y} &= u_{0,y-1}, \quad (y \neq 0) \\ &= u_{0,0}, \quad \text{(by repeated application)} \\ &= 1. \end{aligned}$$

Hence

$$v_0 = \sum_{y=0}^{\infty} p^{-y} = p/(p-1).$$

It now follows that

$$v_x = \left(\frac{p}{p-1}\right)\left(\frac{p+1}{p-1}\right)^x.$$

By the inversion theorem

$$u_{x,y} = \frac{1}{2\pi i}\int p^{y-1}\left(\frac{p}{p-1}\right)\left(\frac{p+1}{p-1}\right)^x dp,$$

taken around a large contour.

The integrand possesses a pole of order $(x+1)$ at $p=1$ and is regular elsewhere and hence

$$u_{x,y} = \frac{1}{x!}\left[\frac{d^x}{dp^x}\{p^y(p+1)^x\}\right]_{p=1}.$$

At A, $x=y=n$ and the required number of paths is consequently

$$\frac{1}{n!}\left[\frac{d^n}{dp^n}\{p^n(p+1)^n\}\right]_{p=1}.$$

But, using Rodrigues' formula,

$$\begin{aligned} \frac{d^n}{dp^n}\{p^n(p+1)^n\} &= \frac{1}{2^n}\frac{d^n}{dx^n}(x^2-1)^n \quad (x=2p+1) \\ &= n!\,P_n(x) \\ &= n!\,P_n(2p+1), \end{aligned}$$

P_n being Legendre's polynomial of order n.

The number of paths is accordingly $P_n(3)$. D. F. L.

1707. Far too eagerly did she [Nancy Milbanke, later Lady Byron] absorb the teaching of the local schoolmaster and therethrough develop a strong taste for mathematics. This had the rather unfortunate effect of inculcating in her a quality of cool appraisement and objective judgment that made her so impossible a wife for a temperamental man.—Una Pope-Hennessy, " The Byrons at Seaham," *Durham Company* (1941), p. 31.

THE CALCULUS REPORT.*

Mr. **K. S. Snell** (Harrow): The production of a calculus report by the Association is meant to stimulate teachers, and to suggest subject matter and methods of presentation. It follows that by the time a report has been in circulation for some months, teachers with ideas are often teeming with criticism about what has been said, in that the report differs from their own ideas, or criticism about what has been omitted. Hence opportunity is now given to you to have your say, and to suggest improvements. Two of those who were responsible for the report are being given first innings, the object being, I assume, to revive information about the report to those of you who have not read it recently; but probably the sooner we leave the field open for you the better, and I anticipate that we may have a more difficult five minutes at the end to attempt to answer.

If the report has value it is due in part to the fact that the main part was written and discussed within a fairly short space of time and reasonable agreement was reached. It is due more to the fact that we had an efficient secretary in Mr. Prag, and more especially to the excellent work of our editor, Mr. Robson, who contrived to bind together individual papers into a connected whole, and to produce compromise editions on topics on which we had disagreement. I am very sorry he is not here to answer questions, as I regard the report largely as his baby, and reckon that he knows more about it than anyone else.

It will be seen that attempts have been made to give different points of view, showing merits and demerits, rather than insistence on one particular method, e.g. in the discussion on the use of differentials. In some of these matters I can now be more dogmatic, and state my own preference, as this is likely to provoke discussion. In scope the report deals especially with the initial presentation of calculus, and the first five chapters at any rate are of direct concern to all teachers of mathematics in Grammar schools, as so many more pupils are now concerned with calculus since the publication of the " alternative syllabus ". In the second course items of special interest have been selected and detailed suggestions about presentation have been made. There is no attempt to cover a complete course.

As I mention topics I will give paragraph references from the report for the benefit of those who are following in their copy.

Section 2 on preparatory work gives suggestions of ideas which can be stressed in elementary work, because the teacher knows these ideas will be wanted in calculus. In graphical work, 2.1, attention can be drawn frequently to the interpretation of gradients, positive and negative. For example in a time-distance graph the steeper the gradient the faster the speed, provided the time axis is always horizontal. A general consideration of steepness naturally leads to the method of measuring speed numerically from the graph. If the graph happens to be a straight line then the speed can be obtained accurately; if it is a curve, how can it be found, and is it possible to calculate it if the equation connecting time and distance is known? Such questions, left unanswered sometimes, stimulate thought in the pupil, and prepare him for subsequent work, provided this is not too far ahead in time. Discussion may arise also as to why the tangent of the angle is used in the definition of gradient, rather than the sine.

Chapter 3 deals with introductory work, needed at the beginning of the calculus course. This needs to be thought out carefully and not hurried. It

* A discussion at the Annual Meeting of the Mathematical Association, January 4, 1952.

has in the past been the custom to precede calculus by a course of analytical geometry, and by a full treatment of various limits. Now we have swung possibly to the opposite extreme, and want to get quickly past the beginning to applications of differentiation. But it is probably well worth while to be slower at the start and have some introductory lessons to ensure full understanding of coordinates and equations of straight lines, and finding the gradient from the equation. This can, of course. be fitted into a course of graphs, so that the pupil is not thinking he is tackling a new subject. At times also during the initial course it pays to do some sketching of curves from their equations. Such work needs much practice in short doses, and a teacher can encourage the two way process, that to every equation there is a curve, and to every curve there can be an equation to represent it, this latter being a harder idea and therefore should be started early.

The initial calculation of gradients is the stumbling block to many pupils, but is not hard if sufficient preparatory work has been done. The report draws attention, 3.2, to the possibility of a kinematical approach, though it does not recommend this as the first start. I have often followed up the measurement of speed from a time distance graph by the calculation of speed at any time, when the equation of the curve is known, and a falling body provides such a suitable equation. I have often found this a most satisfactory first lesson in calculus, partly because speed is something so much more definite to find, and one that has more meaning for a young boy, than gradient in general.

A limit, 3.3 is an interesting idea to a child, and many teachers like to deduce properties of tangents to a circle from corresponding chord properties, by letting the chord move towards a tangent. The other limits here suggested, speed, and limit of $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$ to n terms, as n gets larger, are also interesting and prepare a pupil for the differentiation process. I cannot understand why some want to avoid both the idea and the term limit, when it can be treated informally so easily. 3.4 shows how to obtain the gradient of the curve $y=x^2$ at the point (2,4), by defining the tangent as the one line through (2,4) which is not a secant to the curve. Such a treatment seems to me intelligible to the pupil who has seen the illustration of the secant swinging through the position of the tangent but not otherwise. I should much prefer the calculation to show the limiting process, as in 4.2.

3.43 and 3.44 are interesting additions for the teacher, but of course are not for the pupil at this stage.

Notation, ch. 4, is something which at some time must be tackled. The progress of mathematics has in the past often been retarded by lack of suitable notation, and I think the progress of the pupil can also be retarded in the same way. There are some teachers who maintain that new notation is a hindrance, and try to avoid it altogether in a first course. I think this is a mistake. However, in 4.2 and 4.3 the report shows the first stage of differentiation with differing amounts of notation. My own habit is to call corresponding increments h and k, to obtain the gradient of a chord, i.e. k/h, and then to denote the result of differentiation by $D(y)$ emphasising as soon as possible that differentiation is a process which can represent a gradient or have other meanings. A δ is another piece of notation, which I avoid at first, and in fact I have just finished a first term, with a set of boys of rather low ability, in which I have not used δ. But as consolidation next term I shall start again by differentiating from first principles this time using δx and δy. In this way new notation is gradually introduced, and at the same time the boys will go through the same process in rather new guise, and such repetition of work with a tinge of newness is essential for the stupid boy if he is to gain an understanding. The use of δ seems to me appropriate particularly in finding

the gradient at any point, rather than a particular numerical point. I doubt if I should ever use it as in 4.3. After expressing a gradient as $\frac{\delta y}{\delta x}$ it seems to me that this is the time to call the limiting result $\frac{dy}{dx}$. As this notation is used more and more in scientific and other works it is a pity to allow any student to finish even a very elementary course without meeting the notation. Thus I would ensure that a pupil is introduced, gradually, to D, $\frac{dy}{dx}$, $\frac{\delta y}{\delta x}$. Remember that more than half of those we are thinking of will never get beyond this first elementary course.

Differentials, 4.4, provide a real bone of contention. In the report we give the possibility of their being introduced in the very early stages. Difficulties about this course are also mentioned, and the reader is left to make his own decision as to the best time of introduction.

Chapter 5 on applications does not necessarily suggest an order in which these should be considered. It does stress to the teacher that however interesting are the applications to maxima and minima, there are other important applications. For approximations I like to wait until the δ notation is familiar, and then use the fact that $\frac{\delta y}{\delta x} \simeq \frac{dy}{dx}$, if δx is small. It often happens that this work can be synchronised with a first introduction to the binomial theorem, and it always interests pupils to see from the binomial theorem the degree of accuracy achieved by the δ method, in evaluating for instance $\sqrt[3]{(8{\cdot}13)}$. Rates again, 5.6, provide an excellent application, especially in driving home the $\frac{dx}{dt}$ notation. I like to refer to the " language of calculus ", *i.e.* $\frac{dx}{dt}=2$ is calculus way of saying the speed of an object is 2, in appropriate units, just as in treating easy problems an equation is the algebraic shorthand for a sentence in the problem. I find myself here introducing informally the idea of differentiation of a function of a function. Thus in getting the rate of increase of the volume of an expanding balloon I use $\frac{dv}{dt}=\frac{dv}{dr}\cdot\frac{dr}{dt}$, saying that $\frac{dv}{dr}=3$, for instance, means that the volume is increasing 3 times as fast as the radius, and hence means the same as $\frac{dv}{dt}=3\frac{dr}{dt}$.

In finding maxima and minima, 5.8, the report stresses the value of frequent graphical illustration. No attempt should be made at this stage to bring in second order differentiation, since that would provide a rough and ready rule rather than an understanding of what is being done. Also the boy who is going further with mathematics needs to know the fundamental method of examining the sign of the derivative on either side of the critical value of x.

Chapter 5 may be said to end the first course, apart from the applications of integration to kinematical problems as suggested in 5.1, and to finding areas and volumes of solids of revolution from the results $dA/dx=y$ and $dV/dx=\pi y^2$, or corresponding results with the other axes. These results are established geometrically, as in 7.32, p. 32, and provide an excellent extension of the method of differentiation from first principles. I think the report might at this stage have made a little clearer that this might be considered the end of the first general course.

Chapter 6 then is the first extension for those who wish to go ahead and increase their powers of manipulation, and ability to differentiate more complicated functions. The differentiation of a function of a function causes

difficulty, and must be treated informally at first, and the report advises the checking by the differentiation of functions like $(1+x)^3$, which can also be differentiated after expansion. Those who have been used to an early introduction to differentials may find the method easy to follow, but may also think wrongly that no proof is needed, and this fallacy is dealt with in 6.24. For the benefit of teachers a proof is given in 6.25.

Integration and its applications are grouped together in ch. 7. Stress is laid on the fact that integration can be treated as the limit of a sum and that it was first used in this way by Archimedes. The method is not suitable for those who are doing the most elementary course, but is essential for those who are going on to other applications such as finding positions of centres of gravity and moments of inertia. This does not mean that the integral is to be evaluated by finding a sum and then its limit as $\delta x \to 0$, and so it is essential that the student should understand that the definition of integration as a limit leads to the same result as that of anti-differentiation. A strict proof of this belongs to Analysis and cannot be attempted by a student at this stage, but what is done in 7.21 to 7.33 is enough to convince a normal student, thus preparing him for the more rigorous course if he goes on to it. In 7.35 we show briefly the application of this in finding an area by treating $\lim \Sigma y \,.\, \delta x$ as $\int y \, dx$ between appropriate limits. It is in finding the position of a centre of gravity, or in finding a moment of inertia that a student can find the appropriate integral more easily by thinking of it as the limit of a sum. But it is impossible to be rigorous by that method and so in 7.531 and 7.532 the finding of a centroid is done rigorously by the anti-differentiation method. I would rather use some such method as indicated below, but it is clearly unrigorous to argue from an approximate result to a limit sum, and it was for that reason that the report could not print a method such as I give. I still think, however, that is the kind of method which a student should use, and will understand. If it is put side by side with the rigorous method the student will justify the former by the fact that it produces the correct integral.

This is my solution of 7.532. To find the C.G. of the lamina OAB, of which $NMQP$ is an element of thickness δx. Let ρ, the area density at (x, y), be a

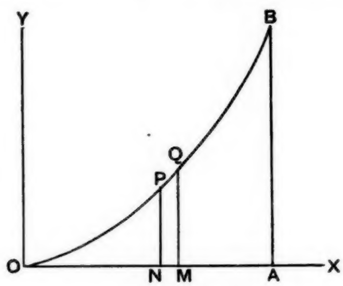


FIG. 1.

function of x. The mass of the element $NMQP \simeq \rho y \, \delta x$. Its moment about $OY \simeq x \,.\, \rho y \, \delta x$. Hence, if $(\bar{x}, \bar{y})$ is the C.G. of the lamina,

$$\bar{x} \int_0^X \rho y \, dx = \int_0^X \rho x y \, dx.$$

Again the C.G. of $NMQP$ is approximately at a height $\frac{1}{2}y$.

Its moment about $OX \simeq \frac{1}{2}y \cdot \rho y\,\delta x$.

$$\bar{y}\int_0^X \rho y\,dx = \int_0^X \tfrac{1}{2}\rho y^2 dx.$$

The use of double and triple integrals, 7.52, makes it possible to write down definitions of centroids of areas and volumes, and their evaluation is much less frightening than is imagined, and I reckon that scientists might well profit by a short course on their use. They would then read scientific books with less awe and more understanding.

Dr. **I. W. Busbridge** (St. Hugh's College, Oxford): Before I proceed to the second half of the report, with Mr. Snell's permission I wish to say something further about the section on volumes and centroids. In our recent Scholarship and Entrance Examination for the Women's Colleges at Oxford, on the elementary mathematics paper for chemists, I set the question "Find the volume of the solid formed by rotating the area of the parabola $y^2 = 4x$, cut off by the line $x = 1$, through 2π radians about the line $x = 1$". Of 31 candidates, 30 attempted it and 20 got it completely wrong. They wrote down $\int \pi y^2 dx$ or $\int \pi x^2 dy$. This is a grave reflection on the teaching of the subject in schools. What interested me more, however, was that of the remaining 10 who got it right, only 3 considered the volume of the element formed by rotating a strip of area $(1-x)\delta y$ about $x = 1$; the rest changed the origin to the point (1, 0). This is a method which we have not considered in the report, but which might well be discussed this afternoon.

Turning to Chapter 8 on the logarithmic and exponential functions, I feel that there is little to say, because the unsatisfactory method of introducing e^x by its infinite series, without an adequate knowledge of convergence on which to base it, has now been dropped. The method most often used nowadays is that in which $\log_e x$ is defined (usually under a pseudonym such as hyp x or ln x) by

$$\int_1^x \frac{dt}{t}.$$

Before the exponential function can be introduced, an assumption must be made, and this should be clearly stated. In order to prove that

$$\log x^a = a \log x, \quad \text{.........(1)}$$

the formula

$$\frac{d}{dx}x^a = ax^{a-1} \quad \text{.........(2)}$$

has to be used. Since (2) has only been established for rational values of a, (1) can only be deduced for rational values of a and we have to *assume* that it is also true for irrational values, an assumption which can be justified very much later by means of rational approximations. This point is emphasised in § 8.13 of the report.

For the scientist, $\int_1^x \frac{dt}{t}$ is not the most natural way of introducing the logarithmic and exponential functions. He meets e^x in connection with laws of growth in which the rate of increase or decrease of a substance is proportional to the amount y of the substance present, so that

$$\frac{dy}{dt} = ky.$$

We deal with this approach in § 8.22. I am not convinced that the method given there is necessarily the best and if anyone has tried a different method, something useful might be said about it this afternoon. Here, again, an

assumption is necessary. Starting from the idea of compound interest, a solution of the form $y=a^t$ is tried. Now a^t has only been defined for rational values of t and we have to assume that it has a meaning for irrational values of t.

Chapter 9 on Expansions is based entirely on the idea of approximations. In practical applications a scientist only uses those terms at the beginning of a series which will make his result correct to so many places of decimals. It is only the mathematician who is interested in infinite series. This chapter gives the scientist all he needs and it forms an excellent basis for the subsequent treatment of infinite series for the mathematician.

The report suggests three ways in which expansions can be obtained . In the first (§ 9.11) we try to write the function $f(x)$ in the form

$$f(x)=a_0+a_1x+\ldots+a_nx^n+x^{n+1}g(x)$$

by means, for example, of long division. If the term $x^{n+1}g(x)$ can be made arbitrarily small by making n large enough, the polynomial $a_0+a_1x+\ldots+a_nx^n$ will give an approximation for $f(x)$ and the magnitude of the error can be estimated.

In the second method (§ 9.2) we get a series of inequalities. For example we can show by integration that

$$x-\frac{x^3}{3!}<\sin x<x-\frac{x^3}{3!}+\frac{x^5}{5!}$$

if $x>0$. Hence the error in taking $x-x^3/3!$ as an approximation for $\sin x$ is less than $x^5/5!$.

Working with the idea of these two methods as a basis, a sub-committee of the Teaching Committee is considering the teaching of convergence. Personally I would exclude all convergence tests from sixth form teaching, basing everything on the limit as $n\to\infty$ of the sum of n terms. If a student learns d'Alembert's ratio test at school, I find that she tends to apply it on all occasions, suitable and unsuitable. Several members of the sub-committee are present and we should be extremely pleased to hear your views on the teaching of convergence.

The third method of obtaining expansions (§ 9.12) is based on the idea of the contact of curves. If two functions $f(x)$ and $g(x)$ are such that

$$f(a)=g(a),\quad f'(a)=g'(a),\ldots,\quad f^{(n)}(a)=g^{(n)}(a),$$

their curves will have nth order contact at the point $x=a$, and each function will give an approximation to the other. When $n=2$, for example, the curves have a common tangent and the same curvature. In particular, if

$$g(x)=f(a)+(x-a)f'(a)+\ldots+(x-a)^nf^{(n)}(a)/n!,$$

the above equations are true and $g(x)$ gives an approximation to $f(x)$. In this method the magnitude of the error must be estimated by some other means until Taylor's theorem is reached.

Chapter 10 on Differential Geometry is one of the best sections of the report. The first part appeared in the *Gazette* a few years ago as an article by Mr. Robson, and it has been reproduced almost unchanged. This part deals with sign conventions, the importance of which are, perhaps, most easily seen in the teaching of dynamics. I always teach sign conventions to my own students in this connection.

Suppose we take as our basic assumption that

$$\frac{ds}{dx}=+\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}\geqslant 0,$$

then we are assuming that $\cos\psi \geqslant 0$ and hence (limiting ourselves to principal values) that

$$-\tfrac{1}{2}\pi \leqslant \psi \leqslant \tfrac{1}{2}\pi.$$

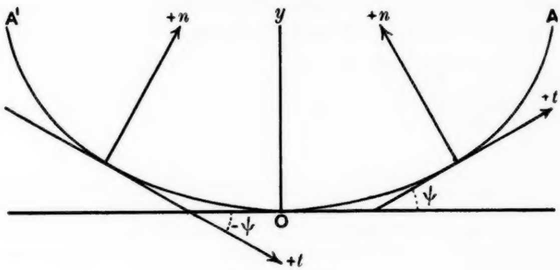


FIG. 2.

Now consider a particle moving on the cycloid $s = 4a \sin\psi$ between the cusps A and A'.

Between O and A, ψ lies between 0 and $\tfrac{1}{2}\pi$; between A' and O, ψ lies between $-\tfrac{1}{2}\pi$ and 0, *not* between $\tfrac{1}{2}\pi$ and π (a mistake often made by students with an imperfect knowledge of sign conventions).

In all problems connected with curvature, sign conventions are of the greatest importance. For example, a particle describing a curve has an acceleration $v^2/|\rho|$ along the *inwards* normal, or v^2/ρ along the *positive* normal. If we use the former for a particle sliding on a sine curve, we have to write down separately the equations of motion for each successive arch of the curve; if we use the latter, one set of equations holds for the whole curve. Details of this example are given in § 10.731.

In § 10.8 we give an excellent treatment of envelopes. This is the only section of the report which assumes a knowledge of partial differentiation. Instead of the conventional treatment, which begins with the intersections of the curves

$$f(x, y, a) = 0, \quad f(x, y, a + \delta a) = 0$$

and which never gives a proper definition of an envelope, we begin with a precise definition of an envelope. Then we show that the coordinates of a point on the envelope of $f(x, y, a) = 0$ *satisfy*

$$f(x, y, a) = 0, \quad \frac{\partial}{\partial a} f(x, y, a) = 0,$$

which are also satisfied by the coordinates of singularities of $f(x, y, a) = 0$. Thus node and cusp loci may also appear, as well as the envelope, from these equations.

I find that I have one minute left in which to deal with Chapter 11 on Differential Equations. The committee was of the opinion that differential equations should be introduced primarily as an aid to applied mathematics; a systematic course should not be given until they have gained an intrinsic interest of their own. The one section about which I must say something is § 11.52. Here we deal with the linear equation with constant coefficients

$$\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + by = 0.$$

The old fashioned method was to try $y=Ae^{mx}$. This led to the quadratic equation

$$m^2+2am+b=0$$

with roots m_1 and m_2. If $m_1 \neq m_2$ you asserted that

$$y=Ae^{m_1x}+Be^{m_2x}.$$

The case $m_1=m_2$ was usually deferred for several pages. My own reaction to this method was "Why try $y=Ae^{mx}$?" In § 11.52 we give a method of solution which does not depend on trial, and in § 11.53 we give the parallel method for the difference equation

$$u_n+2au_{n-1}+bu_{n-2}=0.$$

I feel that these are some of the most important sections of the report. In connection with this chapter there is one thing which I should myself like to know. At what stage of the sixth form have you found it possible to introduce a systematic course on differential equations and in what examinations (if any) should questions on it appear?

I propose to say nothing about Chapter 12 on History. This was written by Mr. Robson and Mr. Prag. Mr. Prag is present and he is prepared to answer any questions connected with Chapter 12.

Mr. **A. J. Moakes** (St. Paul's) thought that the Report did not sufficiently cater for the needs of the science pupil, particularly in the introduction of the logarithm and exponential. The scientist would need infinite series, which had been ignored in the Report.

Mr. **J. E. Blamey** (Wood Green) advocated a return to Napier's ideas in dealing with the logarithm.

Mr. **C. O. Tuckey** (formerly Charterhouse) pleaded for some work on infinite series.

Mr. **V. I. Todhunter** (Dartmouth) supported the method for e^x given in paragraph 8.22 of the Report.

Mr. **C. T. Daltry** (Institute of Education) said that the problem was largely one of how much to teach. He wondered how much should be said about irrationals, and what use should be made of kinematics. He drew attention to the valuable book by Carl Boyer, *Concepts of the Calculus*.

Mr. **Campbell** thought that the omission of electrical applications would diminish the value of the Report to teachers in technical colleges.

Mr. **W. J. Langford** (Battersea Grammar School) said that infinite series would be discussed in the Report on Sixth Form analysis now in preparation.

Mr. **J. T. Combridge** (King's College, London) pointed out that technical applications are to be given in the Association's *Compendium*, and asked for suggestions.

Mr. **W. O. Storer** (Highgate) wondered if it would be possible to abandon the notation dy/dx.

Dr. **Busbridge** said that some discussion on irrationals would appear in the Analysis Report, and Mr. **Snell** replied to Mr. Moakes' criticism by reminding members that in many schools mathematicians and scientists were taught together and a common course must be provided. He pointed out to Mr. Campbell that electrical applications would not be intelligible to pupils on the arts side.

1708. It is reported of the State of Indiana that it was only narrowly stopped from passing a law to say that the mathematical symbol Pi should be equal to 4 instead of to 3·14159 recurring, because it was easier.—*News-Chronicle*, October 14, 1950. [From many correspondents; see No. 1703.]

MATHEMATICAL NOTES.

2292. *On a type of plane algebraic curve of degree n with a multiple point of order* $(n-1)$.

The type to be considered is that in which the tangents to the n-ic at the $(n-1)$-ple point O are all distinct, and the conics of closest contact to two real branches at O have $(n+2)$-point contact.

Let the tangents to the two conics at O meet them again at A and B. Then with ABO as triangle of reference and suitable homogeneous coordinates the equations of the two conics may be put into the form

$$yz=x^2+pxy, \quad xz=y^2+qxy. \qquad \text{(i)}$$

We shall suppose for the present that $pq(p^3-q^3)(pq-1)\neq 0$. Let the equation of the n-ic be

$$zxy(a_0x^{n-3}+a_1x^{n-4}y+\ldots+a_{n-3}y^{n-3})=b_0x^n+b_1x^{n-1}y+\ldots+b_ny^n. \qquad \text{(ii)}$$

The conics have $(n+1)$-point contact if

$$a_0=b_0,\quad pa_0+a_1=b_1,\quad \ldots,\quad pa_{n-4}+a_{n-3}=b_{n-3},\quad pa_{n-3}=b_{n-2};$$
$$a_{n-3}=b_n,\quad qa_{n-3}+a_{n-4}=b_{n-1},\quad \ldots,\quad qa_1+a_0=b_3,\quad qa_0=b_2. \qquad \text{(iii)}$$

These give on elimination of $b_2, b_3, \ldots, b_{n-2}$, if we suppose $a_{-1}=a_{n-2}=0$,

$$a_{s+3}+pa_{s+2}-qa_{s+1}-a_s=0 \qquad \text{(iv)}$$

for $s=-1, 0, 1, 2, \ldots, (n-5)$.

This linear finite-difference equation leads to

$$a_{r-1}=A\begin{vmatrix}\alpha^{n-1} & \beta^{n-1} & \gamma^{n-1}\\ \alpha^r & \beta^r & \gamma^r\\ 1 & 1 & 1\end{vmatrix} \div \begin{vmatrix}\alpha^2 & \beta^2 & \gamma^2\\ \alpha & \beta & \gamma\\ 1 & 1 & 1\end{vmatrix}, \qquad \text{(v)}$$

where A is independent of r and α, β, γ are the values of t given by

$$t^3+pt^2-qt-1=0,$$

so that $x=\alpha y$, $x=\beta y$, $x=\gamma y$ are the lines joining O to the remaining intersections of the conics (i). The equation of the n-ic follows from (iii).

The conics have $(n+2)$-point contact if $b_1=b_{n-1}=0$, that is

$$pa_0+a_1=qa_{n-3}+a_{n-4}=0. \qquad \text{(vi)}$$

Using (v), we have

$$\begin{vmatrix}\alpha^n & \beta^n & \gamma^n\\ \alpha & \beta & \gamma\\ 1 & 1 & 1\end{vmatrix} \div \begin{vmatrix}\alpha^2 & \beta^2 & \gamma^2\\ \alpha & \beta & \gamma\\ 1 & 1 & 1\end{vmatrix}=0, \quad \begin{vmatrix}\alpha^n & \beta^n & \gamma^n\\ \alpha^{n-1} & \beta^{n-1} & \gamma^{n-1}\\ 1 & 1 & 1\end{vmatrix} \div \begin{vmatrix}\alpha^2 & \beta^2 & \gamma^2\\ \alpha & \beta & \gamma\\ 1 & 1 & 1\end{vmatrix}=0 \qquad \text{(vii)}$$

as two equations to determine p and q.

We may express these equations explicitly in terms of p and q by elimination of $a_0, a_1, a_2, \ldots$ from (iv) and (vi) or otherwise. For the sake of clarity the result is written down for the case $n=7$, but the method is general. If the two conics have $(n+1)$-contact, the tangents at O are $xy=0$ and

$$\begin{vmatrix} y^4 & xy^3 & x^2y^2 & x^3y & x^4\\ p & -q & -1 & 0 & 0\\ 1 & p & -q & -1 & 0\\ 0 & 1 & p & -q & -1\\ 0 & 0 & 1 & p & -q \end{vmatrix}=0;$$

and they have $(n+2)$-point contact if

$$\begin{vmatrix} p & -q & -1 & 0 & 0 \\ 1 & p & -q & -1 & 0 \\ 0 & 1 & p & -q & -1 \\ 0 & 0 & 1 & p & -q \\ 0 & 0 & 0 & 1 & p \end{vmatrix} = 0, \qquad \text{(viii)}$$

and a similar equation with p and q interchanged.

If we write $k \equiv pq$, $t \equiv p^3$, equation (viii) is readily transformed into

$$\Delta_5 \equiv \begin{vmatrix} 1 & -k & -1 & 0 & 0 \\ 1 & t & -k & -t & 0 \\ 0 & 1 & 1 & -k & -1 \\ 0 & 0 & 1 & t & -k \\ 0 & 0 & 0 & 1 & t \end{vmatrix} = 0. \qquad \text{(ix)}$$

Expanding the last determinant, we see that

$$\Delta_{2m} = t\Delta_{2m-1} + k\Delta_{2m-2} - t\Delta_{2m-3}, \quad \Delta_{2m+1} = \Delta_{2m} + k\Delta_{2m-1} - \Delta_{2m-2}.$$

Thence, by induction, (ix) can be written when $n = 2m$ or $n = 2m+1$

$$t^m + f(n-2, 1)t^{m-1} + f(n-4, 2)t^{m-2} + f(n-6, 3)t^{m-3} + \ldots = 0, \qquad \text{(x)}$$

where $f(s, r)$ is the coefficient of x^s in $(k-x)^r/(1-x)^{r+1}$.

The equation (x) in t has two roots, p^3 and q^3, whose product is k^3. Therefore, an equation to give k and thence p, q can be obtained for any given value of n by routine processes.

It remains to remove the restrictions imposed so far. Firstly, if $pq = k = 1$, $f(s, r) = 1$. In this case, (x) does not lead to real curves.

Next, suppose that $p = q$. In (v) we may take

$$\alpha = \cos 2\theta + i \sin 2\theta, \quad \beta = \cos 2\theta - i \sin 2\theta, \quad \gamma = 1,$$

and putting $u_n = \sin n\theta \operatorname{cosec} \theta$, we have

$$p = -u_3\,; \quad a_0 : a_1 : a_2 : \ldots : a_{n-3} = u_{n-3}u_3 : u_{n-4}u_3 : \ldots : u_1 u_{n-2}.$$

If the conics have $(n+2)$-point contact, (vii) gives $u_n = 0$, so that θ is a multiple of π/n. In this we have excluded $p = q = 1$, $p = q = -3$ which give $\alpha = \beta$. But it is at once seen that the conics cannot have $(n+2)$-point contact for these values of p and q.

Now, let $p = q = 0$. It can be shown at once that no non-degenerate n-ic exists having $(n+2)$-point contact with $yz = x^2$, $xz = y^2$ at the $(n-1)$-ple point $(0, 0, 1)$ unless n is a multiple of 3, when the curve is

$$zxy(x^{n-3} + x^{n-6}y^3 + x^{n-9}y^6 + \ldots + y^{n-3}) = x^n + x^{n-3}y^3 + x^{n-6}y^6 + \ldots + y^n.$$

The possibility of $p \neq 0$, $q = 0$ is excluded. For, if when $q = 0$, we denote the determinant in (viii) by d_5 and the corresponding determinant with p and q interchanged by d_5', we have $d_n = pd_{n-1} - d_{n-3}$, $d'_n = pd'_{n-2} - d'_{n-3}$. These give, by induction,

$$d_n = p^n - {}^{n-2}C_1 p^{n-3} + {}^{n-4}C_2 p^{n-6} - \ldots,$$

$$d'_{2m} = p^m + {}^{m-1}C_2 p^{m-3} + {}^{m-2}C_4 p^{m-6} + \ldots,$$

$$d'_{2m+1} = -{}^{m}C_1 p^{m-1} - {}^{m-1}C_3 p^{m-4} - {}^{m-2}C_5 p^{m-7} - \ldots.$$

Since d_n cannot vanish for any negative value of p, and d'_n cannot vanish for any positive value, we cannot have $d_n = d'_n = 0$ for a real value of p.

The case $n=4$ is of special interest. We have :

If the conics of closest contact to two branches of a quartic curve at a triple point have six-point contact, so has the osculating conic of the third branch ; and the curve is projectable into one whose polar equation is $r=a\cos 3\theta$.

HAROLD SIMPSON.

2293. *On a special type of plane trinodal quartic curve.*

This note contains a brief discussion of the plane quartic curve with three nodes, one of which is a crunode (with real tangents) at which both conics of closest contact have seven-point contact.

It is readily seen that the equations of a rational quartic with a real crunode can be put into the form

$$x:y:z=t(\pm k+t):t^2(1+kt):(a+bt^2+k^2t^4) \qquad \text{.....................(i)}$$

by a suitable choice of parameter t, triangle of reference, and homogeneous coordinates. In this, $y=0$ and $x=0$ are the tangents at the crunode, the corresponding values of t being 0 and ∞. The line $z=0$ is chosen to make the coefficients of t and t^3 zero in the last bracket of (i). In what follows, the positive sign is taken in the ambiguity. The lower sign may be treated in a similar manner, but does not only give rise to a real curve.

Put now

$$X\equiv -y-ux, \quad Y\equiv y+vx,$$

$$(u-v)Z\equiv z+(u+v+2uv)x+(u+v+2)y, \qquad \text{..................(ii)}$$

where u and v are the values of θ given by

$$k^2(k^2-1)\theta^2-(k^2+k^2b+a)\theta+(k^2-1)a=0. \qquad \text{........................(iii)}$$

If $k^2=1$, the quartic (i) has a triple point at (0, 0, 1). It may be shown that, if (iii) has equal roots, (i) has a tacnode. Excluding these two cases (i) becomes

$$x:y:z=t(k+t):t^2(1+kt):\{k^2uv+[(k^2-1)(u+v)-uv-1]t+k^2t^4\} \quad \text{...(iv)}$$

and from (ii) and (iv) on elimination of t

$$(k^2-1)X^2Y^2+vZ^2X^2+uZ^2Y^2+(1-v)X^2YZ+(1-u)XY^2Z$$
$$+(u+v)XYZ^2=0, \qquad \text{.........(v)}$$

so that $X=0$, $Y=0$, $Z=0$ are the lines joining the three nodes in pairs.

It is easily verified that the conics of closest contact with (iv) at (0, 0, 1) are

$$yz=uvx^2+(k^2-2)uvxy+\{(k^2-1)(u+v-k^2uv)-1\}y^2,$$

$$xz=y^2+(k^2-2)yx+\{(k^2-1)(u+v-k^2)-uv\}x^2$$

and that they meet the quartic again respectively where

$$0=\{(k^2-1)(u+v-2k^2uv+uv)-1\}+\{(k^2-1)(u+v-k^2uv)-2\}kt-k^2t^2,$$

$$0=\{(k^2-1)(u+v-2k^2+1)-uv\}t^2+\{(k^2-1)(u+v-k^2)-2uv\}kt-k^2uv.$$

Both the conics have six-point contact at (0, 0, 1) if

$$(k^2-1)(u+v-2k^2uv+uv)=1, \quad (k^2-1)(u+v-2k^2+1)=uv.$$

Excluding for the present $2k^2=3$, we get from these

$$uv=1, \quad (k^2-1)(u+v)=2k^4-3k^2+2.$$

The conics have seven-point contact if also

$$(k^2-1)(u+v-k^2uv)=2,$$

which gives $k^2=2$, $u+v=4$, $uv=1$.

With these values of k^2, $u+v$, uv, equation (v) becomes, on replacing X, Y, Z by $x/(1-v)$, $y/(1-u)$, $-\frac{1}{2}z$,

$$y^2z^2+z^2x^2+2x^2y^2+2xyz(x+y-2z)=0. \quad\text{(vi)}$$

We deduce that, if the two osculating conics at a crunode of a trinodal quartic curve both have seven-point contact, the curve can be projected so as to have an axis of symmetry, the other two nodes are acnodes, the four real and two unreal inflexional tangents touch a conic which also touches the two lines joining the crunode to the acnodes, etc.

Putting $x=r\exp(\theta i)$, $y=r\exp(-\theta i)$, $z=a$ in (vi) we see that the curve with an acnode at which two unreal conics have seven-point contact is projectable into the curve with polar equation

$$(r+a\cos\theta)^2=a^2(3-\cos^2\theta).$$

Taking $2k^2=3$, we get the trinodal quartic with a crunode at which one osculating conic has seven-point contact and the other has six-point contact. It is given by putting $2k^2=3$, $u+v=uv=-2$ in (v). It has two crunodes and an acnode.

It can be shown in a similar manner that the equation of a trinodal quartic, one of whose nodes is a flecnode, can be put into the form

$$vy^2z^2+uz^2x^2+x^2y^2+x^2yz+xy^2z+(u+v)xyz^2=0.$$

The osculating conic at the flecnode has six-point contact if $2uv+1=u+v$. It has seven-point contact if $u+v=3$, $uv=1$, and then the other two nodes are acnodes. HAROLD SIMPSON.

2294. *The quotient of two quadratic functions.*

In Notes 1429, 1457, and 2049, Dr. R. F. Muirhead, Mr. R. Holmes, and Mr. A. A. K. Ayyangar have discussed the construction of rational functions of the form

$$(ax^2+bx+c)/(a'x^2+b'x+c')$$

with given maximum and minimum values β and α, $\beta>\alpha$. The last note asks that the discussion should not be restricted to a few numerical cases, but should deal with a standard form. In what follows I give a method which seems simpler than any given before, and I also deal with the question of standard forms. These, it will be proved below, are

$$(\beta-y)/(y-\alpha)=k(x-B)^2/(x-A)^2, \quad\text{(1)}$$

$$(\beta-y)/(y-\alpha)=k(x-B)^2, \quad\text{(2)}$$

$$(\beta-y)/(y-\alpha)=k/(x-A)^2, \quad\text{(3)}$$

$$1/(y-\alpha)=k(x-B)^2/(x-A)^2, \quad\text{(4)}$$

$$1/(y-\alpha)=k(x-B)^2, \quad\text{(5)}$$

$$1/(y-\alpha)=k/(x-A)^2. \quad\text{(6)}$$

If all the symbols represent real numbers, and if k is positive, the right-hand sides of these equations cannot be negative. Hence in (1), (2) and (3), $\beta\geqslant y\geqslant\alpha$, while in (4), (5) and (6), $+\infty\geqslant y\geqslant\alpha$. If, however, k is negative, the corresponding results are that y cannot lie between β and α, and that y cannot lie between $+\infty$ and α (or, alternatively, that $\alpha\geqslant y\geqslant-\infty$).

As a numerical example, use formula (1) with $\alpha=2$, $\beta=5$, $A=-1$, $B=2$, $k=2$, giving for a case in which $5\geqslant y\geqslant 2$,

$$(5-y)/(y-2)=2(x-2)^2/(x+1)^2,$$

so
$$(5-y)/3=(2x^2-8x+8)/(3x^2-6x+9)$$

and $y=5-(2x^2-8x+8)/(x^2-2x+3)=(3x^2-2x+7)/(x^2-2x+3)$.

The amount of work here seems less than that required for the same final result in Note 1429.

We shall now prove that the above six standard forms are the only ones which can occur. In general, if

$$y = (ax^2 + bx + c)/(a'x^2 + b'x + c'),$$

to each value of y, real or complex, correspond two values of x, which are equal if and only if

$$(b'y - b)^2 = 4(a'y - a)(c'y - c).$$

Denote the roots of this equation by α and β, and the corresponding values of x by A and B. The condition for the equality of α and β can be reduced to

$$(ac' - a'c)^2 = (bc' - b'c)(ab' - a'b),$$

which is also the condition that the two quadratic functions of x should have a common factor involving x, *i.e.* that y should be merely the quotient of two *linear* functions of x. Excluding this degenerate case, we see $\alpha \neq \beta$, and so $A \neq B$. Now $(\beta - y)/(y - \alpha)$ must be the quotient of two quadratic functions of x, and $y = \beta$ must give $x = B$ (twice); similarly $y = \alpha$ must give $x = A$ (twice). Hence, if α, β, A, B are all finite, the only possibility is formula (1).

It is convenient, in view of what follows, to write this in a different form. If $x = C$ gives $y = \gamma$, $(\beta - \gamma)/(\gamma - \alpha) = k(C - B)^2/(C - A)^2$, so (1) can be replaced by

$$\{(\beta - y)(\gamma - \alpha)\}/\{(y - \alpha)(\beta - \gamma)\} = \{(x - B)(C - A)\}^2/\{(x - A)(C - B)\}^2.$$

Taking the limit when $A \to \pm\infty$, the right-hand side becomes $(x - B)^2/(C - B)^2$, so giving a formula of the form (2), though with a different value of k. Similarly (3) is obtained from $B \to \pm\infty$, (4) from $\beta \to \infty$, (5) from $\beta = \infty$ and $A \to \pm\infty$, and (6) from $\beta \to \infty$ and $B \to \pm\infty$. The results of putting $\alpha \to -\infty$ are not essentially different from those obtained from $\beta = +\infty$, so need not be added to the list of standard forms.

These results were obtained by a rather different method, and from a different point of view (that of the complex variable) in a paper by Miss M. N. Strain and myself (The Conformal Transformation

$$Z = (lz^2 + 2mz + n)/(pz^2 + 2qz + r),$$

J. Lond. Math. Soc., Vol. 22, 1947, pp. 165–7.) In this paper any of the symbols could denote complex numbers, so the condition $\beta > \alpha$ did not arise. As there was then no distinction between β and α, formulae (2) and (3) became equivalent, reducing the number of standard forms to five. The geometrical interpretation for the complex variable is of some interest as a method of attacking a type of problem often set in the Mathematical Tripos and other examinations.

H. T. H. PIAGGIO.

2295. *Evaluation of π.*

From Gregory's series

$$\pi = 4\{1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \tfrac{1}{13} - \dots\}$$

$$= 4\{1 - \tfrac{2}{15} - \tfrac{2}{63} - \tfrac{2}{143} \dots\}$$

$$= 4 - \tfrac{1}{2}\left\{\frac{1}{1 - \frac{1}{16}} + \frac{1}{2^2 - \frac{1}{16}} + \frac{1}{3^2 - \frac{1}{16}} + \dots\right\}$$

$$= 4 - \tfrac{1}{2}\left\{\sum_{1}^{m} \frac{1}{n^2 - \frac{1}{16}} + \sum_{m+1}^{\infty} \frac{1}{n^2 - \frac{1}{16}}\right\}.$$

This may also be deduced from the known result

$$\cot z = \frac{1}{z} + 2z\Sigma \frac{1}{z^2 - n^2\pi^2}$$

by putting $$z = \tfrac{1}{4}\pi.$$

Now

$$\frac{n-\frac{1}{2}}{n^2-n+\frac{5}{16}} - \frac{n+\frac{1}{2}}{n^2+n+\frac{5}{16}} < \frac{1}{n^2-\frac{1}{16}}$$

$$< \frac{n^2-n+\frac{1}{2}}{(n-\frac{1}{2})(n^2-n+\frac{9}{16})} - \frac{n^2+n+\frac{1}{2}}{(n+\frac{1}{2})(n^2+n+\frac{9}{16})}.$$

Replace n by $m+1, m+2, \ldots$ to infinity and add. Then

$$\frac{m+\frac{1}{2}}{m^2+m+\frac{5}{16}} < \sum_{m+1}^{\infty} \frac{1}{n^2-\frac{1}{16}} < \frac{m^2+m+\frac{1}{2}}{(m+\frac{1}{2})(m^2+m+\frac{9}{16})}.$$

Hence

$$4 - \tfrac{1}{2}\left\{\sum_{1}^{m} \frac{1}{n^2-\frac{1}{16}} + \frac{m+\frac{1}{2}}{m^2+m+\frac{5}{16}}\right\} > \pi$$

$$> 4 - \tfrac{1}{2}\left\{\sum_{1}^{m} \frac{1}{n^2-\frac{1}{16}} + \frac{m^2+m+\frac{1}{2}}{(m+\frac{1}{2})(m^2+m+\frac{9}{16})}\right\}.$$

With $m=1$, this gives slightly closer bounds than those of Archimedes, viz. $3\frac{1}{7}$ and $3\frac{10}{71}$; $m=2$ gives $\pi > 3\frac{15}{106}$; $m=3$ gives $\pi < 3{\cdot}1416$ and $m=6$ gives $\pi < 3\frac{16}{113}$.

N. M. Gibbins.

2296. *The prime line and a generalisation to n dimensions.*

1. In Note 2132 (May, 1950) Yuan-Jen Roan shows that if two variable points P, Q be taken on a fixed line u, and if p, q be their polars with respect to a fixed non-singular conic Γ, then the perpendicular from P on to q meets the perpendicular from Q on to p at a point on a fixed line v, the "prime line" of u.

In fact, if K is the pole of u with respect to Γ, then v is the polar of K with respect to the director circle of Γ. From this it immediately follows that the relationship between u and v is a line collineation when the director circle is not degenerate, that is, provided Γ is not a parabola or rectangular hyperbola.

2. A generalisation to n-dimensional euclidean space is as follows.

Theorem

Let Γ be a non-singular central quadric primal, and let Σ be its director n-sphere, that is, the locus of points through which there pass n mutually perpendicular tangent primes to Γ. Let π be a fixed prime, K its pole with respect to Γ, σ the polar prime of K with respect to Σ, and let P_i $(i=1, 2, \ldots, n)$ be n linearly independent points on π. Let α_1 be the prime through P_1 perpendicular to the polar line with respect to Γ of the secundum through $P_2P_3 \ldots P_n$. Let the primes $\alpha_2, \alpha_3, \ldots, \alpha_n$ be similarly defined. Then, for varying P_i, the locus of the point of intersection of these n primes α_i is the prime σ.

Proof. Let the equation of Γ referred to its principal axes be

$$\sum_{i=1}^{n} x_i^2/a_i^2 = 1.$$

(It is *not* assumed that $a_i^2 > 0$.)

Then Σ has the equation

$$\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} a_i^2.$$

Let K be the point $(k_1, k_2, \ldots, k_n)$. Let p_{ij} be the jth coordinate of the point P_i, and let P_{ij} be the co-factor of p_{ij} in the determinant $\Delta = | p_{ij} | \neq 0$.

Let

$$S_j = \sum_{i=1}^{n} P_{ij}.$$

Then π has the equation

$$\sum_{i=1}^{n} S_i x_i = \Delta,$$

and as the polar of K with respect to Γ it has the equation

$$\sum_{i=1}^{n} k_i x_i / a_i^2 = 1.$$

Comparing coefficients,

$$k_i = a_i^2 S_i / \Delta. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Now σ, the polar of K with respect to Σ, has the equation

$$\sum_{i=1}^{n} k_i x_i = \sum_{i=1}^{n} a_i^2,$$

that is, from (i),

$$\sum_{i=1}^{n} a_i^2 S_i x_i = \Delta \sum_{i=1}^{n} a_i^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Now the polar line h_1 of the secundum through $P_2 P_3 \ldots P_n$ (being the meet of the polar primes of these $n-1$ points) will have direction ratios $l_1, l_2, \ldots, l_n$ given by

$$\sum_{j=1}^{n} l_j p_{ij} / a_j^2 = 0 \quad (i = 2, 3, 4, \ldots, n),$$

because, for instance, the normal to the polar prime of P_2 has direction ratios p_{2j}/a_j^2 and is perpendicular to h_1. Solving these $n-1$ equations in the ratios l_i we have at once

$$l_i = \rho a_i^2 P_{1i}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

where ρ is independent of i and non-zero.

Since α_1 is the prime through P_1 whose normal has these direction ratios, it has the equation

$$\sum_{j=1}^{n} l_j (x_j - p_{1j}) = 0,$$

that is, from (iii),

$$\sum_{j=1}^{n} a_j^2 P_{1j} x_j = \sum_{j=1}^{n} a_j^2 P_{1j} p_{1j}.$$

Similarly, the other $n-1$ primes α_i have equations

$$\sum_{j=1}^{n} a_j^2 P_{ij} x_j = \sum_{j=1}^{n} a_j^2 P_{ij} p_{ij}. \quad (i = 2, 3, \ldots, n).$$

It is now evident that on adding together the equations of these n primes α_i we obtain

$$\sum_{j=1}^{n} a_j^2 S_j x_j = \Delta \sum_{j=1}^{n} a_j^2,$$

which, from (ii), is the equation of σ.

Thus, we see that a point on all the primes α_i is also on σ, and the result follows.

C. F. Fisher.

REVIEWS.

Vorlesungen über Zahlentheorie. By H. Hasse Pp. xii, 474. DM. 42: geb. DM. 45. 1950. Grundlehren der mathematischen Wissenschaften, 59 (Springer, Berlin)

This admirable book reminds one immediately of the classical works with the same title written by other distinguished mathematicians such as those by Dirichlet (with Dedekind's additions), Kronecker and Landau. These suggest that by now a tradition has been established that such a title should be used only for really important treatises produced by great masters. The present volume conforms to this view and is a most worthy successor to its predecessors. It deals with what may be called the classical theory of numbers, that is, essentially those parts of the subject which have been known for many years and which have served as a basis for further studies. The book has two objects. One is to familiarize the reader with concrete results and to present the material in the way in which it was discovered and has grown. The other is to prepare the reader for considering the subject from a more general and abstract point of view and relating it to recent progress. The author's choice of material has been dictated by the great importance he attaches to the natural numbers and statements about them as the primary objects of number-theory. He then considers the developments arising from this point of view. He is convinced that in this way, more interest and importance will attach to results in number-theory. He also takes the opportunity of giving a number of results which have not found their way into the usual books or now are more frequently omitted.

The book consists of four main sections:

(1) the foundations,
(2) quadratic residues,
(3) Dirichlet's prime number theorem,
(4) quadratic fields.

The first begins with prime numbers, the factorisation of numbers and the fundamental theorem of elementary arithmetic. These are presented in such a way as to introduce the reader to abstract concepts and to familiarize him with them. Thus in discussing the greatest common divisor, he introduces ideals of integers. Linear congruences, mod m, are dealt with in the simplest and most general point of view by considering the residue classes mod m as rings and then their decomposition into a direct sum of other rings of residue classes. These concepts will be very useful to the student later on. The section also contains the usual results about Euler's ϕ function, the Moebius inversion formula, etc. The author stimulates the readers' interest by introducing a few unsolved, conjectural, results. There is the question of the existence of an infinity of Fermat prime numbers of the form $N=2^{2^n}+1$. Many of these numbers are known to be composite. Thus when $n=73$, N is composite and divisible by $5.2^{75}+1$. There is also an account of Artin's conjecture that for every non-square integer $w \neq \pm 1$ there is an infinity of primes p such that w is a primitive root of p.

The section on quadratic residues shows features common to Hasse's books. There is the careful attention to detail and skilful handling which enables him to deal with a multiplicity of particular cases in not too unpleasant a manner. There is the discussion of the different aspects of fundamental results, for instance, the criteria for the quadratic characters a mod p. He considers the significance of results such as the law of quadratic reciprocity for the Legendre symbol $\left(\frac{a}{p}\right)$ as well as the congruence properties of the symbol

considered with respect to both its numerator and denominator. The familiar elementary proof of the law is then followed by one depending upon Gauss' sums, and so a short account of the relevant properties of the cyclotomic field $P(\exp 2\pi i/p)$ over the rational field P is given. He then deals with the more general Legendre-Jacobi symbol and also with the Kronecker symbol.

The section ends with an account of a topic which unexpectedly has proved to be of the greatest interest and importance in recent years, namely, the evaluation of the number N of solutions of a congruence

$$f(x, y) \equiv 0 \pmod{p},$$

where p is a prime number and $f(x,y)$ is a polynomial with integer coefficients. It is almost unbelievable to what abstruse considerations this question has led, for example, algebraic function fields and Riemann hypotheses for the zeta-functions associated with them. When $f(x,y)$ is of the second degree, the results have long been known. It is then quite easy to find N and nothing essentially new arises. When, however, $f(x,y)$ is of degree greater than two, the question becomes really interesting and deep results exist. Thus if

$$f(x,y) = y^2 - ax^3 - bx^2 - cx - d,$$

then, in general,

$$|N - p| < 2\sqrt{p}.$$

It is impossible to give a proof without exceedingly lengthy developments except in some simple instances dating from Gauss. Thus when $p \equiv 1 \pmod 4$ and $f(x,y) = y^2 - x^3 - x$, N can be expressed very simply in terms of the representation of p as a sum of two integer squares. Similarly when $y^2 \equiv x^3 - 1 \pmod p$, and $p \equiv 1 \pmod 3$, the representation of p in the form $p = a^2 + 3b^2$ comes in.

There is however one result familiar to Hasse with which the section might well have ended. A conjecture of Artin's proved by Chevalley states that if the congruence

$$f(x_1, x_2, \ldots, x_n) \equiv 0 \pmod{p},$$

where f is a polynomial of degree r with integer coefficients has one solution, then if $n > r$ it has another solution : for example,

$$a_1x_1^3 + \ldots + a_4x_4^3 \equiv 0 \pmod{p}$$

has a solution other than $x_1 \equiv x_2 \equiv x_3 \equiv x_4 \equiv 0 \pmod p$.

The third section deals exhaustively with Dirichlet's prime number theorem, namely, that if r is prime to m, the arithmetic progression $r + mx$ contains an infinity of primes when $x = 0,1,2,\ldots$. Elementary proofs are well known in some simple cases such as $4x - 1$, $6x - 1$, and are based upon the proof due to Euclid of the existence of an infinity of primes. A number of such results follow from the theorem proved here, namely, that if a is not a square, there is an infinity of primes p such that $\left(\frac{a}{p}\right) = 1$, and also an infinity such that $\left(\frac{a}{p}\right) = -1$; e.g., for the progressions $4x + 1$, $6x + 1$. Further results, including $mx - 1$, are found by considering cyclotomic polynomials and their properties.

The method for the general result depends upon the use of analysis as introduced by Dirichlet into number-theory. It generalises ideas and results of Euler originating with his identity

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

where $s > 1$ and p runs through all the primes. Now Abelian groups and their characters enter essentially into the argument. Hasse's treatment of charact-

ers is singularly elegant. The characters $\chi(a)$ arise by considering the residues of a (mod m). What, however, is specially important is to consider whether and how this character can be defined with respect to moduli other than m. His discussion makes very pleasant reading in a domain where often other writers seem to be involved in details whose significance is not clear. Then new zeta functions are formed by introducing characters in the general term of the Riemann zeta function and are defined by

$$\zeta(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}.$$

Limiting processes have to be considered in which $s\to 1$, and the whole crux of the proof is to show that $\zeta(1,\chi)\neq 0$. The usual elementary analytical proof is given, and then another depending upon the simple analytical properties of a function defined by a Dirichlet's series. Finally an algebraic-number-theoretic proof is given depending upon the imbedding of a quadratic field in a cyclotomic field. To express the result in its simplest form, a short digression on the theory of algebraic numbers is necessary. He gives a concise but lucid account of the seemingly mysterious entities called divisors, so that he can introduce briefly the zeta function of an algebraic field.

The last section deals with quadratic fields, written as, say, $K=P(\sqrt{d})$ of discriminant d. The units of the field are considered in great detail. Continued fractions are introduced rather elegantly and applied to the representation of quadratic surds and of the units. The method of approach to the arithmetic in quadratic fields is based upon ideas introduced by Kummer in his discovery of ideal numbers in cyclotomic fields, but is presented in the more abstract and modern form. Hasse introduces the divisor theory by considering in a quadratic field the structure of the rings formed by the residue classes taken with respect to a rational prime p. Thus when $\left(\frac{d}{p}\right)=1$, this ring is isomorphic to the direct sum of two representations of the residue classes of the rational integers mod p. He is thus able to give a very satisfying and complete account of the divisors in a quadratic field and to show that the divisors form an Abelian group with unique factorisation in terms of prime divisors. It is then easy to see how the divisors are related to ideals and to discuss the laws of factorisation in quadratic fields. The whole treatment is very lucid and elegant. The number of classes of divisors (or of ideals) is found as usual by considering the residue of the zeta function of the field at $s=1$. The standard result for $d>0$ does not give the class number in a rational form; but for d a prime, Hasse gives his own investigation expressing the result rationally.

He relates the quadratic field to the law of quadratic reciprocity and shows the significance of this considered from the point of view of class field theory. The section ends with a systematic treatment of the general Gauss's sums. He has had occasion to use them in different parts of the book. He now treats them more generally and gives an account of their relation to the Lagrangian resolvents associated with a cyclotomic field. He ends with an account of Kummer's conjecture on the roots of the cubic resolvent, and the corresponding one for the quartic resolvent.

Fortunate is the student of number-theory who comes across this splendid book at an early stage of his studies. Its influence upon his mathematical education is certain to be considerable and may well be decisive. What a wealth of ideas and storehouse of results he will be introduced to! This instructive and comprehensive book will solve for him the important and difficult problem of aligning and harmonising the old classical knowledge and treatment with the new; and it will impress upon him the fundamental and vital role played by some of the now more usual abstract concepts. I need

only mention again the prominence given to the groups defined by sets of residues and their relation to the Galois theory of some of the equations naturally arising; and also to the structural aspects of results and theorems. Emphasis throughout is given to ideas, and also to indications of extensions, generalisations and wider points of view. The book is remarkable for the clearness, and pleasing nature, of many of its proofs. Some of them, however, occasionally require careful reading, dealing as they do with abstract considerations of some generality. The sooner in his career that the student is plunged into the midst of them, the better for his later mathematical peace of mind; and it will be very difficult to find a more pleasant book to dive into. Finally, the book will be most useful for the formation of mathematical taste and an appreciation of what matters and is significant and vital in number-theory.

For many years, those interested in number-theory have been under great obligations to Professor Hasse for the many books he has written. The present one will add considerably to these obligations, and readers will be ever grateful to him for the great service he has done them in producing this excellent book. L. J. MORDELL.

Integral Transforms in Mathematical Physics. By C. J. TRANTER. Pp. ix, 118. 6s. 1951. (Methuen Monograph on Physical Subjects)

This book is to be regarded as a companion volume to J. C. Jaeger's *Introduction to the Laplace Transformation*, which appeared in the same series two years ago. However, the present volume deals with the transforms $\bar{f}(p)$ of functions $f(x)$ defined by

$$\bar{f}(p)=\int_a^b f(x)K(x, p)dx,$$

for the choices $b=\infty$ and

$$\begin{array}{lll} a=0, & K(x,p)=\exp(-px) & \text{(Laplace Transform)},\\ a=0, & =\sin(px),\ .\cos(px) & \text{(Fourier Transform)},\\ a=-\infty, & =\exp(ipx) & \text{(complex Fourier Transform)},\\ a=0, & =xJ_n(px) & \text{(Hankel Transform)},\\ a=0, & =x^{p-1} & \text{(Mellin Transform)}. \end{array}$$

J_n is here a Bessel function of the first type. For each class of transforms a few examples from mathematical physics are worked out. There are also chapters on the numerical evaluation of integrals, and on the use of transforms when a and b are finite. Most of the examples selected for detailed attention come from the theory of elasticity and the theory of the conduction of heat, and it is a distinct merit of the book that some of the examples are worked out, using different transforms.

The assumptions made by the author regarding his readers appear to be (i) that they are not expert mathematicians, and the subject is therefore developed *ab initio*, and in an elementary manner, and (ii) that they wish to acquire familiarity with *all* the transforms cited. However, these two assumptions will often be mutually exclusive. The beginner will want occasionally to lean back and, glancing at the distant landscape, select one valley for attention rather than another. The author gives him little opportunity for doing so. In a future edition it would appear desirable, therefore, to include some remarks regarding the applications of the transforms which are not discussed in the book, and to give a rather more detailed comparison between the various transforms and their range of application.

There is a good list of books at the end of the volume, but the list of original papers, which owing to lack of space cannot be exhaustive, is also not very

representative. It may be worthwhile adding a brief comment about the method of selection in a future edition, or else refer the reader to one of the more complete lists of references in the larger textbooks.

P. T. LANDSBERG.

Éléments de Mathématiques. By N. BOURBAKI. XII. Livre IV, Fonctions d'une variable réelle (théorie elémentaire) : Ch. 4, Equations différentielles ; Ch. 5, Étude locale des fonctions ; Ch. 6, Développements Tayloriens généralisés, Formule sommatoire d'Euler-Maclaurin ; Ch. 7, La fonction gamma. Pp. 200. 1951. (Actualités scientifiques et industrielles, 1132 ; Hermann, Paris)

Chapter 4 of this instalment of the Bourbaki treatise consists of two sections on differential equations, while the remaining three chapters bear little relation to chapter 4 but are linked together by the idea of asymptotic expansions in terms of functions of a scale at infinity.

The first section of chapter 4 is concerned with existence theorems for equations of the form

$$\frac{dx}{dt}=f(t, x), \qquad \text{(1)}$$

where $f(t, x)$ takes its values in a complete normed vector space E, being defined in a product set $I \times H$, where I is an interval contained in the space $\mathbf{R}$ of real numbers and H is an open sub-set of E. The general outline of the method followed, that of obtaining a solution as the limit of a uniformly convergent sequence of " solutions approchées ϵ près ", will be familiar to a generation brought up on the famous *Cours d'Analyse* of de la Vallée Poussin, but the treatment is of course entirely abstract and there are many generalizations and refinements which repay careful study. It should perhaps also be mentioned that one advantage of the abstract treatment is that there is no need for separate consideration of a system of simultaneous equations

$$\frac{dx_i}{dt}=f_i(x_1, x_2, x_3, \dots, x_n) \quad (i=1, 2, 3, \dots, n),$$

the corresponding existence theorems being obtained at once by taking E to be a product space. The second section of the chapter is on linear equations ; that is, on equations of the form (1) where $f(t, x)$ is defined in $I \times E$ and $f(t, x)-f(t, 0)$ is, for each fixed t of I, additive and continuous in E.

Chapter 5 contains important and fascinating extensions and generalisations of the ideas of the " Infinitärcalcül " of P. du Bois-Reymond which were put on a rigorous foundation by G. H. Hardy in the well-known Cambridge tract *Orders of Infinity*. The range of the topics treated is indicated by the sectional headings : (i) Comparaison des fonctions dans un ensemble filtré, (ii) Développements asymptotiques, (iii) Développements asymptotiques des fonctions d'une variable réelle, (iv) Application aux séries à termes positifs. The functions considered in the first three sections take their values in general in normed vector spaces, although certain results are restricted to the special case of real-valued functions. There is also a notable appendix, entitled *Corps de Hardy*, which is concerned with the generalisation of the L-functions of the Cambridge tract.

In chapter 6 a generalised Taylor expansion,

$$f^{(p)}(x+h)=\sum_{m=0}^{n} \frac{1}{m!} u_m(x) U_h^{\xi} \{f^{(m)}(\xi)\}+R_n(x, h),$$

is obtained under appropriate conditions. In this expansion $U=D^p V$ is an

operator permutable with the operator D of differentiation, $u^m(x) = V^{-1}(x^m)$ is an Appell polynomial of index m, $U_h{}^\xi\{g(\xi)\}$ denotes the value of $U(g)$ at the point h and

$$R_n(x, h) = -U_h{}^\xi \left\{ \int_0^{\xi - x - h} \frac{1}{n!} u_n(x+\eta) f^{(n+1)}(\xi - \eta) d\eta \right\}.$$

It is noted in passing that, if $U(g) = g$, the expansion reduces to the ordinary Taylor expansion, but the important application considered is to the case

$$U(g) = (e^D - 1)g.$$

With this determination of the operator U, the corresponding Appell polynomial $u_m(x)$ is the Bernoulli polynomial $B_m(x)$ and the Bernoulli number b_m is defined as $B_m(0)$. Properties of the Bernoulli polynomials and numbers are then obtained and the Euler-Maclaurin summation formula is deduced very easily from the generalised Taylor expansion.

Finally, chapter 7 contains a short and elegant account of the gamma function. Following the treatment originated by Bohr and Mollerup and developed by Artin, $\Gamma(x)$ is defined for $x > 0$ as the (unique) solution of the functional equation

$$f(x+1) = xf(x),$$

which is such that $\log f(x)$ is convex and $f(1) = 1$. The usual properties are easily deduced, the range of definition is extended to the complex plane and the asymptotic expansion obtained.

The instalment is characterised throughout by extreme elegance and precision, and new insight into well-known topics is obtained by the introduction of ideas of algebra and topology. This does mean, however, that the reader whose knowledge of these subjects is limited has to spend considerable time in looking up relevant definitions and explanations in earlier books of the treatise. W. L. C. S.

Special Functions of Mathematical Physics. By W. MAGNUS and F. OBERHETTINGER. Translated from the German by J. WERMER. Pp. 172. \$3.50. 1949. (Chelsea, New York)

The second German edition of this work was reviewed on p. 267, Vol. XXXII. Of the contents of that edition, the present translation lacks the appendices containing Fourier expansions and summation formulae, and thus appears to have been made from the earlier German edition. The difference between the editions may be typified by the chapter on cylinder functions, which is doubled in the later German version, and that on Integral Transforms which is substantially unchanged. This translation is, nevertheless, a useful and reasonably comprehensive collection of formulae.

The present edition claims to have corrected a number of errors ; this must be set against a more haphazard layout in the printing. In particular, the conditions for the validity of formulae show a distressing tendency to stray away from their owners. Caution is needed, for instance in a case such as this : p. 1 ends with an analytic expression for $\Gamma(z)$. The first line of p. 2 is a fresh sentence " Where n is the greatest non-negative integer smaller than $Re(-z)$ ", followed by an expression involving a product from $n = 1$ to infinity. Other doubts, usually soluble by reference to their context, are as to whether a condition refers to a group of formulae or only to its nearest neighbour. R. B. H.

Vorlesungen über Fouriersche Integrale. By S. BOCHNER. (Rep.) Pp. 229. \$3.95. 1948. (Chelsea Co., New York)

This reprint of a well-known book should be welcomed by a large class of readers. The declared aim of the author has been to obtain as many different

kinds of properties as possible in a comparatively short space, particular attention being given to those which are useful for applications. Thus there are sections on the Fourier-Stieltjes integral and positive definite functions, on difference-differential equations, on the Laplace integral and on functions of more than one variable, as well as on standard theories for functions of $L(-\infty, \infty)$ and $L^2(-\infty, \infty)$. Moreover, the section on positive definite functions concludes with an elegant and enlightening proof of Parseval's Theorem for almost periodic functions.

The book is very readable and the treatment elementary and self-consistent, only well-known properties of the Lebesgue integral being needed. These are stated and explained in an appendix at the end of the book. There is one unusual definition which calls for remark : a function is said to be differentiable in an interval if it coincides with an indefinite Lebesgue integral at all points of the interval. Only one rather unnoticeable reference is given, in the text, to this definition, and the reader who misses it is naturally considerably puzzled. The theory is illustrated by a large selection of examples.

W. L. C. S.

Introduction to Hilbert Space and the Theory of Spectral Multiplicity. By Paul R. Halmos. Pp. 114. \$3.25. 1951. (Chelsea Publishing Company, New York)

The main purpose of this book is to make available in English the theory of the unitary invariants of normal operators in Hilbert Space, and the last of its three chapters is devoted to this subject. The first two contain an excellently set out account of Hilbert Space and of the bounded operators, projectors and bounded hermitian and normal operators on it. The development of the theory of these last operators is made to rest on measure theory : as a result, and by making full use of the latest work on the subject, the author gives a treatment in which the geometry of the space plays a large part.

The problem of unitary invariants is the problem of finding criteria that two operators can be transformed into one another by an isomorphism of the space. For finite normal operators the solution involves the characteristic values of the operators, and the multiplicities with which they occur. For a separable Hilbert space a solution was given by Hellinger, and greatly simplified by Friedrichs, who showed that it involves the multiplicities with which closed sets of the plane occur in the spectrum. For non-separable spaces the problem is much more complicated, and was first solved by Wecken. The third chapter of the present book sets out a solution equivalent to Wecken's but more geometrical in treatment.

The book can be recommended as an introduction to Hilbert Space theory, for a certain class of readers. Its merits are its accuracy, its concise and lively style and its concern to give the reader an idea of the general direction of the arguments. However, it is clearly written for readers with a training in the abstract modes of mathematical reasoning and, particularly, in measure theory. It would not be very suitable for readers interested in physical applications of the theory, partly because these applications involve only separable spaces, but mainly because no mention is made of unbounded operators and consequently of differential operators. Moreover, there is a lack of concrete examples which may be felt even by trained mathematicians ; this is particularly so in the last chapter, which, inevitably, involves long chains of abstract reasoning and in which an example to show the relationships between the different classes of projectors involved would have been very useful to the reader, as would an indication of the simplifications possible in the separable case.

These minor points should not obscure the fact that Professor Halmos has

written a stimulating and useful book. The volume is well and accurately produced, and there seem to be few errors. A few misprints may be worth recording :

P. 15, theorem 2, the square sign should be omitted ;
P. 53, line 13, $\| A - \lambda_0 x \|$ should read $\| (A - \lambda_0)x \|$;
P. 86, line 10 from below, " column " should read " row ".

J. L. B. COOPER.

Linear Computations. By PAUL S. DWYER. Pp. xi, 344. 52s. 1951. (John Wiley, New York ; Chapman and Hall)

This book presents a fairly extensive and detailed discussion of such basic linear problems as the solution of simultaneous linear equations, the evaluation of determinants, and the calculation of the adjoint, inverse and characteristic equation of a matrix. The emphasis is on methods suitable for use with a desk calculating machine, and plenty of exercises are provided, an essential part of any text on numerical processes. Each chapter begins with a summary of its contents, and if several methods are available for the same problem, their relative advantages and disadvantages are given. Most of the book is easy to read, and the arrangement of material is such as to make fundamental techniques intelligible to readers " who know the basic facts of high school algebra ". There is a lengthy study of the accumulation of errors, and a shorter account of some statistical applications. Both authors and subjects are adequately indexed, although important references might conveniently have been printed in heavy type.

Professor Dwyer has written an admirable book, and the criticisms which follow are not of the contents, but relate to what is omitted. In a text of 335 pages, iterative methods receive only $1\frac{1}{2}$ pages, mainly references ; they are excluded " because the direct methods, either exact or approximate, seem satisfactory ". This reason fails to exclude the inferior methods of row and diagonal division for solving linear equations, and the implication that indirect methods are unnecessary cannot be accepted. In fact, many statistical problems lead to a set of equations with large diagonal coefficients, where only the solution is required and not the inverse matrix. An iterative process is then ideal.

It is also surprising that a Wiley Publication in Statistics should limit the discussion of errors to deterministic formulae, and give no relevant probability statements. Thus the error of a mean of 1000 heights (p. 322) may indeed have a maximum of 0·50 inches, but with random discrepancies it is most unlikely to exceed 0·03 inches.

R. L. PLACKETT.

Differential Equations. By H. B. PHILLIPS. 3rd edition, revised. Pp. 149. 24s. 1951. (Wiley, U.S.A. ; Chapman and Hall)

The author is Professor Emeritus at the M.I.T. and it is clear that he has the technical student in mind. Examples for the reader are classed either as *Exercises*, in which given D.E.'s are to be solved, or as *Problems*, requiring the setting up of D.E.'s for solution. It is a pity that in the Exercises all coefficients are numerical ; by introducing literal coefficients and equations from standard theory of pure and applied science the author could have formed a bridge between the Exercises and the Problems.

There are four chapters : (I) D.E.'s of First Order with Variables Separable ; (II) Other Equations of First Order ; (III) Special Types of Second Order Equations ; (IV) Linear Equations with Constant Coefficients. The book is, in fact, too brief for the purposes of the latest syllabuses for degrees in science and engineering. These include such topics as solution in series, normal modes, and the solution in Fourier series of partial D.E.'s. There is, too, an increasing

demand from technologists for instruction in the method of transforms, so that there is room for a new presentation of Differential Equations to meet the needs of technical students. From this point of view, Prof. Phillips' book is somewhat " vieux jeu ". C. G. P.

School Mathematics, Part III. By H. E. PARR. Pp. viii, 237 (with tables), xxii (answers). 6s. 6d. Without answers, 6s. 1951. (Bell)

Only 108 pages of this final book of the series are devoted to new work. The rest consists of 65 test papers, carefully selected and arranged, to offer a full three terms' revision.

Nearly all the new work consists of items one expects to come at the end of the course. It concludes with good chapters on " The Derived Function " and " Integration ". It is good to see " Civic Arithmetic " and " Navigation " included, though some parts of these would make excellent earlier work. The Calculus chapters lack a little Coordinate Geometry for at least one of the G.C.E. examinations.

A long period of revision is necessary before a public examination and it is a novel and interesting idea to provide for it as in this book. If the course suits your choice from the various G.C.E.'s, you are made independent of books of papers or collections of examples.

In the attempt to establish the Unified Course idea, this is an important book. Already it is clear that the idea is not proving as helpful as at one time seemed certain. In the earlier stages the various branches do not seem to lose their identities and all we get is chapter mixing. Some authors are already daring to declare against unification and some are trying partial unification. Those who want to try a Unified Course book should consider this one. They must face a probable upheaval in order of syllabus which most books of this type demand. H. B.

Gewöhnliche Differentialgleichungen. By G. HOHEISEL. 4th edition. Pp. 129. DM. 2.40. 1951. Sammlung Göschen, 920. (Walter de Gruyter, Berlin)

It is good to see the famous Göschen series again, now indeed in paper covers, but at no undue increase in price. Hoheisel's book is an admirable specimen of the clear exposition, concentrated information and up-to-date point of view we expect from books of this series. In the first 80 pages we have the methods of integration and the existence and uniqueness theorems for ordinary differential equations, compactly set out and illustrated. The remainder of the text provides a brief but most helpful introduction to boundary-value and related problems, eigenvalues, the Sturm-Liouville theory, distribution of zeroes. There are no examples for the reader, but No. 1059 in the same series gives Hoheisel's own collection for this purpose. T. A. A. B.

Handbuch der Laplace-Transformation, Bd. I : Theorie der Laplace-Transformation. By G. DOETSCH. Pp. 581. Sw. fr. 74 : bound, Sw. fr. 78. 1950. (Verlag Birkhäuser, Basle)

In intention and execution this may be considered as a revision, involving complete rewriting and a doubling of length, of the first three parts of the author's well-known *Theorie und Anwendungen der Laplace-Transformation* (Springer, 1935). A second volume, dealing with applications, is promised.

The first section of the book consists of three chapters dealing with the definition and general analytic properties of the Laplace Transform. An improvement over the earlier book is that the integral involved is allowed to be a Lebesgue integral. The regions of holomorphism, of ordinary and of absolute convergence and their relationships are discussed, and the effect on the transform of various operations on the generating function are worked out,

including a detailed discussion of the convolution of two functions. The second section deals with inversion formulae; almost exclusively, and at great length, with the complex inversion formula: for example the effects of deformation of the path of integration and of differentiation and integration under the integral sign are pursued in detail. The third section discusses what is called a generalised Laplace Transform—in fact, the result of summing the integral involved by Cesáro means. The fourth section treats the transforms of particular classes of functions. The fifth and final section deals with Abelian and Tauberian theorems, asynptotic expansions and so on. It follows the treatment in the earlier book but has a number of additions. An appendix gives a summary of results from real variable theory used in the book. This is followed by notes on the text, a list of books and references, and an index.

If a " Handbuch " should be a comprehensive and all-round account of its subject, this book has certain notable gaps, apart, of course, from the matters left to the second volume. The author's attention is almost entirely focussed on questions lying inside the classical theory of functions of a complex variable, and on this aspect of the theory of the Laplace Transform he is very thorough and complete. Questions of the real-variable theory, however, receive little discussion: the moment problem is omitted because, according to the introduction, it belongs to the field of the Laplace-Stieltjes transform, which is barely discussed. The treatment of Tauberian theorems is along classical lines: Wiener's general Tauberian theorem is stated, but not proved. As a result, there is singularly little overlapping with Widder's book.

The book has two main preoccupations. The first is to give the main formal properties of the Laplace Transform in the complex plane, particularly those which occur in applications to the solution of functional equations. The second is with the properties of the transforms as a class of complex functions, and especially with finding analogies with the properties of the class represented by Taylor series about the origin. However, the simple results which hold for Taylor series have no complete analogues: regions of holomorphism and convergence are not so tidily related, and only partial results can be given. Some work of Doetsch, first published here, aims at getting more rounded results by using Cesáro means to extend the domain of definition of the transform, or by varying the path of integration.

The great merits of the book are its thoroughness, accuracy and attention to detail. Very little is left to the reader to fill in. As the other side of the medal it has defects of longwindedness and pedantry and a certain narrowness of outlook. They are illustrated by the naming of the Cesáro means of the Laplace Transform a generalised transform: and by the denial (Note 196, p. 558) to Paley and Wiener of a proof of a famous theorem of theirs, whereas the theorem in question follows by very simple arguments from those which they prove in detail. Although there are some remarks about metric function spaces, there is little of the spirit of modern functional analysis; an example of this is the description (p. 452) of the various cases of Parseval's theorem as being only " in äusserlicher Analogie " to one another; duality theory has shown their inner connection.

For all this, the book will prove invaluable as a reference book in its field. The printing and production are beautifully and accurately done.

J. L. B. Cooper.

Die Zweidimensionale Laplace-Transformation. By D. Voelker and G. Doetsch. Pp. 259. 43 fr. bound, 39 fr. unbound. 1950. (Verlag Birkhäuser, Basle)

This book gives an introduction to the use of the two variable Laplace Transform for the solution of boundary value problems. An account of the

transforms considered as a function of two complex variables is given in the first chapter. This is very brief, involving only a discussion of the domain of convergence under fairly restrictive assumptions on the functions, which suffice for applications, and an account of the effect on the transform of linear substitutions, differentiation and convolution of the generating function. The second section, four chapters long, deals with the applications to the solution of boundary value problems for two independent variables in a quadrant of the plane: certain special boundary value problems, of the heat conduction equation, the wave equation and Poisson's equation are discussed, and this is followed by a systematic account of the general second order equation with constant coefficients, and by a discussion of systems of first order equations. The sixth chapter deals with applications to differential equations in three variables, with two of them varying over a quarter-plane, and a final chapter gives some applications to deriving functional relationships. The second part of the book, about 100 pages long, consists of a list of correspondences of functions and transforms due to the first-named author. These lists include not only pairs of correspondences of particular functions and transforms, of the type familiar for the one-variable transform, but also a list, of half the total length, showing the effects on the transforms of certain general operations on the generating functions. The usefulness of this list derives from the fact that the solution of a partial differential equation appears as the result of an operation on the initial values.

The theoretical part of the book is very much subordinated to the applications: for example, the question of the inversion of the transform is nowhere discussed in general. The whole book is written with an eye on the "Praktiker" who will use its prescribed sets of rules and tables of correspondences to solve his problems, and it is very well worked out from this point of view. The "Praktiker" is urged not to mistrust the Lebesgue integral because he is not familiar with it, as he is with the Riemann integral; so that he must be a more sophisticated mathematician than his British counterpart, who has rarely heard of either. The attitude to the use of the method for the solution of equations is semi-heuristic: the general formulae are proved only under certain restrictive assumptions which may not hold in all cases in applications, but if their use leads to a solution, this can be tested directly for its validity.

From a more general point of view the treatment has certain weaknesses: there is a tendency to suit the problems chosen to the method, instead of varying the method to suit the problem. For example, there is no indication of how one should treat problems where the two variables vary over, say, a strip rather than a quadrant. Again, in most discussions of many-variable problems two is simply the most convenient instance of plurality; but here it is a fixed number. Thus, while the logical parallel to handling two variable problems by the two-dimensional transform would be to use three-dimensional transforms for three variable problems, this is not done. For example, the problem of heat conduction in an infinite plate of finite width, treated in § 26, could be handled more simply by the three variable transform, or, better still, either by a two-dimensional Laplace transform combined with a sine transform over the width of the plate or by a double sine transform over the plate. Unfortunately the book gives no hint of these possibilities, and does not even mention the connections with multiple Fourier transforms, whose use would frequently avoid the need to introduce extra boundary conditions and to eliminate them by conditions of compatibility, as is necessary in the method followed in the book. In fairness to the authors, one may say that their procedure has the practical advantage of enabling the user to stay, whenever it can be applied, within the limits of the table of correspondences which is provided.

This book is the first in any language dealing with the applications of the two-dimensional Laplace transform, and it will be found very useful both for its guidance in the use of the transform and for its table of correspondences. It is very well printed and produced. J. L. B. C.

Statistische Methoden für Naturwissenschafter, Mediziner und Ingenieure. By A. LINDER. Second edition. Pp. 238. Sw. fr. 30 (cloth boards); 26 (paper). 1951. (Birkhäuser, Basel)

The author of this introductory volume on the theory and applications of statistical methods is Professor of Mathematical Statistics in the University of Geneva. The first edition appeared in 1945 ; in the second edition the bibliography has been brought up-to-date but only a few minor amendments have been made to the text.

The book consists of a brief introduction and four chapters. The first three chapters describe clearly and simply the computation of the usual statistics of frequency distributions, tests of their significance, and the analysis of variance. These chapters are arranged so that the practical reader can quickly find the formula he needs ; he will find it unencumbered by theory and aptly illustrated by a fully-worked numerical example. The necessary tables are provided.

The fourth and last chapter, on sampling theory, is more difficult and more interesting. It contains a concise account of the statistical theory applied in the earlier chapters. The author acknowledges his debt to R. A. Fisher, P. C. Mahalanobis, and especially to M. G. Kendall's treatise *The Advanced Theory of Statistics*. The mathematical treatment is elegant, but few English doctors or engineers could jump comfortably from the arithmetic of the first three chapters to the n-dimensional spaces of the fourth.

It is pleasing to see a text-book of statistics printed in German. Up to the present time, statistical methods have aroused comparatively little interest in Germany and Professor Linder's book should help to make them more widely known there. Except for its conciseness the book has little to offer those who can read English, since several works of similar scope in English are readily available at a lower cost. B. C. B.

Aerodynamics of Supersonic Flight (An Introduction). By ALEXANDER POPE. Pp. xi, 185. 25s. 1950. (Pitman Publication Corporation : New York, Toronto, London)

This book is designed to cover an elementary one-term course for undergraduates, presumably in the Georgia Institute of Technology, the name of which Institution appears on the title page. The stated aim of the author is to introduce such students to problems in the field of supersonic flow, and to help them to decide whether or not they wish to continue their study of the subject at a more advanced level. The book is well printed on good paper and is lavishly illustrated.

The account covers a very wide field ; it starts with some of the physical concepts fundamental to a study of supersonic motion and then proceeds through aspects of flow in a duct, supersonic flow from nozzles and round a corner, oblique and normal shocks, to theories of flow in which mathematical approximations are made, to a descriptive account of some American supersonic tunnels and finally to descriptions of present-day tentative theories about supersonic aeroplanes.

No great demand is made upon the mathematical knowledge of the reader, but owing to the nature of the subject the formulae in the text soon appear to become complex. In this connection two-and-a-half pages devoted to an explanation of " Abbreviations and Symbols " are a great help. In certain

places the author has introduced some apparently complicated system of double and triple suffixes, but there is some logic in his proposals.

This is a highly individualistic book in which the author, who is obviously an enthusiast for his subject, has made his own synthesis of topics. The style is " American " and vital, but the " English " chosen will, in many places, jar upon the susceptibilities of those who try not only to train their students in precision of thought, but also to instil precision in the use of the English language.

Speaking generally, this book contains a mixture of descriptive Engineering and elementary Mathematics, probably more suitable for use in American Colleges of Technology than in England where, up to now, " Aerodynamics of Supersonic Flight " has been considered to be an " advanced " subject suitable for postgraduate study and higher technology, rather than a " basic " one suitable for undergraduate study. L. R.

Hydraulic Transients. By George R. Rich. Pp. x, 260. 51s. 1951. (McGraw-Hill)

This latest volume in the series of " Engineering Societies Monographs " is written by one of America's pre-eminent designers of major hydraulic projects. It is largely a reflection of his experience in the analysis of water hammer and allied phenomena of wave propagation, with non-linear damping. The author gives his unqualified support to the method of tabular integration so widely used in this field, for investigating the behaviour of such systems when subjected to various input disturbances. The method is presented in detail and amplified at considerable length by design calculations drawn from American projects.

Sufficient material is introduced to solve only the immediate problem considered without recourse to specialised methods and techniques. Turbine regulation and governing are discussed for example, insofar as they affect the motion of the gate valve in hydro-electric installations. The stability investigations are amplified by vector diagrams and all the necessary conclusions drawn, without direct reference to servo-mechanism theory.

The design calculations for surge tanks are considered at length, with particular emphasis on the Johnson Type of Differential Surge Tank. Certain aspects of wave propagation in open channels are examined and the filling of navigation locks is also discussed. The graphical method of water hammer analysis, based on the concept of wave propagation, apparently finds little favour with the author and is relegated to the last chapter.

The book is kept within reasonable compass by omitting any reference to model studies and contains only passing references to design details.

For those interested in the control of water hammer and associated problems, it covers those aspects of design which are concerned with the theoretical prediction of system performance. It may, however, also be studied with profit by all those who are interested in the method, or the application of tabular integration to the type of equations encountered in this field.

D. A. Jobson.

The River Mathematics. By A. Hooper. Pp. 370. 18s. 6d. 1951. (Oliver and Boyd)

The Main Stream of Mathematics. By Edna E. Kramer. Pp. xii, 321. 30s. 1951. (O.U.P., London : Geoffrey Cumberlege)

These two books with similar titles treat of the development of mathematics from its earliest beginnings to modern times, but in very different manners.

It is difficult to know just for whom Mr. Hooper's book is intended, and the problem is not simplified by the absence of a preface. There are many refer-

ences to the general reader who " may skip the next ten (or twenty) pages ", and internal evidence suggests that much of the book is expanded from the author's wartime lectures to intending pilots and navigators. Mostly it consists of snippets of history sandwiched between large chunks of text-bookwork, which often admittedly appeals to intuition, gives no formal proof and relies on dogmatic statement. There are many worked examples, and in two cases a set of exercises for the student. Eight pages treat of the manipulation of common fractions; the addition of directed numbers is well done, but the treatment of the other rules is sketchy; the explanation of the slide-rule lapses into unintelligibility, having suffered cuts without revision. The best feature of the book is its development of number-scales, from positive integers through common and decimal fractions and irrationals to complex numbers. Vectors, conics and calculus are briefly treated and the book ends with tables of circular functions and radians (but not logarithms) and an index. In Fig. 109 and others there are too many arrow-heads; three printer's errors have escaped notice and there is some clumsy English. The author is undecided whether to use " cosec " or " csc ".

Miss Kramer's book is written for the layman who is intelligent enough to want to know something of relativity and rockets. Discursive at times, yet scholarly, this remarkable book does not disdain to call on legend and imagination; thus the charming but tragic romance of Lilavati leads through lotus flowers and pollywogs to the full development of Arabic numerals, while the curious methods adopted by Lewis Carroll in the preparation of a Euclid lesson open up an outline of Hilbert's and other postulational systems. That stimulating third chapter begins with the identification of Mr. Chad with the spirit of things lacking and goes on to the ideas of abstraction, generalisation, the variable; it shows that the commutative law does not hold in cake-making and makes a brief excursion into mathematical philosophy with some easily understood examples of Boolean algebra: it ends with a reference to the golden rectangle and Birkhoff's formula for aesthetic appeal. We see something of the " Helen of geometry " (the cycloid), of the curious repetition of the logarithmic spiral in nature, of " the most glamorous woman mathematician " (Sonya Kovalevsky) " that ever lived ", of how von Neumann's paper on Poker affected military strategy. Cybernetics, relativity, Galois groups—the layman is led into deep waters here, and Miss Kramer does not always attempt to rescue him. The final chapter shows how Cantor's theory of the Infinite affords one method of escape from Zeno's paradoxes.

In writing a book of this kind it is difficult to know how much knowledge the reader may be assumed to possess. Thus, our layman must not be frightened by the use, without definition, of words like " nomogram " or " proton ", though he knows little or no trigonometry. But this is a minor point: the student of mathematics, as well as the layman, will find much to interest him.

Errors are few, as one might expect from the Oxford Press. I find " practiced " (p. 10), " consructed " (p. 224), a decimal point omitted in Fig. 59 and a faulty alignment in " sin θ " on p. 144. B. A. S.

The Teaching of Mathematics in Post-Primary Schools. By J. H. Murdoch. New Zealand Council for Educational Research. London: Geoffrey Cumberlege. 12s. 6d.

The author, who is responsible for training teachers in New Zealand, has written this book for the New Zealand Council for Educational Research. The educational set-up in New Zealand differs from that in England so that what the author has to say is not applicable in its entirety to conditions in England where we have the tripartite system of secondary education. In New Zealand

it would appear that secondary education is based on the comprehensive school system. Mr. Murdoch has his feet planted firmly on the ground. He hits out at those people who adopt glib theories about which they have given little thought. Unfortunately these people are many and their plausible opinions are easily swallowed by others. The author raps over the knuckles those who apply the utility test for the content of mathematical teaching. As these do not apply the same test to any other subject, their attitude is quite untenable. They would have little other than the four rules taught to many children on the theory that mathematics has no " transference " value, a theory that rests on no experimental foundation. On the basis of his 15 years of experience of graduate students, Mr. Murdoch considers that the mathematics students form a more evenly capable group than any other. This is interesting because I have before me a cutting from the A.M.A. of November 1932. In this, W. H. D. Rouse, who had 30 years experience with Cambridge post-graduate students of every faculty, says that mathematical and classical scholars were in a class by themselves in their power of being able to get to grips with an entirely new subject (Sanscrit). Again, Frank Sandon, writing in the A.M.A. of July 1939, stated that factorial analysis indicated that mathematics and Latin are the subjects most highly saturated with " g ", *i.e.* with high intellectual capacity and quickness of seizing the relatedness of things and ideas. These facts force us to the conclusion that a mathematical education may have values other than utilitarian.

Mr. Murdoch does not go into details of teaching specific topics. Nevertheless, he has much to say that is valuable. He refers to the " core " syllabus which is studied by all children however weak. This syllabus deals only with essential elements. In 1949, 52% of New Zealand entrants to training colleges had done no mathematics beyond the core standard. (20% in 1947, 40% in 1948.) In South Africa, where a similar weakness exists, the students averaged 50·6% in a test on the four rules ; the standard V average was 73·4%. Yet many of these students would be expected to teach mathematics to, among others, future teachers of mathematics in the primary schools. Here we have a vicious circle and the author points out the dangers of the neglect of this subject in the training colleges. One or two themes keep recurring in the book with variations as in a symphony. " Complete mastery of the four fundamental processes is essential." " Children enjoy, above all, the feeling of complete mastery." " Accuracy means practice." " The alleged drudgery may exist for the teacher rather than for the child." " It is possible to live in this modern world with little knowledge of history, geography, art, etc., but a mastery of elementary calculation is essential for every citizen."

The author is concerned about the danger that the gifted child may not receive adequate attention—a danger which is by no means absent. He quotes W. D. Reeve of U.S.A. who wrote in *The Mathematics Teacher* : " The most retarded pupil in the secondary school today is the gifted pupil, the one with the scholarly mind. The secondary school machine is geared to turn out a mediocre product."

To the teacher of average and weak pupils the author says, " The teacher must at all times inspire self-confidence." " It is better that your pupils retain their confidence and limited power than that they flounder in difficulties." " The first year work must be unhurried." He advises the head of the department to take at least one core class (a different one each year) for some years. Only thus will he be able to give practical advice to junior members of the mathematical staff. There are many heads of departments here who fail to carry out this valuable advice.

The author is in favour of incorporating some elementary statistics in the syllabus. He is interested in the way mathematics is taught in technical

schools where the subject is brought into contact with industry and practical needs and he thinks that the methods used in these schools provide valuable lessons for other schools. He makes frequent references to the reports issued by the Mathematical Association and he also quotes from reports of other countries. In this connection I would offer criticism on two points. It is unfortunate that he quotes several times from the Spens Report. Articles and letters in the A.M.A. have pointed out many contradictory statements which this report makes on mathematics. The Report fails to appreciate the value and purpose of mathematics and its recommendations, if carried out, would do untold harm to the teaching of this subject. On the other hand, Mr. Murdoch makes only passing reference to the Jeffrey Report. Yet this Report has touched off a revolution in the teaching of mathematics in English grammar schools.

At the end of the book there is a valuable bibliography which includes an excellent list of books suitable for the mathematics library. S. I.

Einführung in die analytische Geometrie. By L. Bieberbach. 4th edition. Pp. 168. DM. 8.90. 1950. (Verlag für Wissenschaft, Bielefeld)

This is a quite elementary textbook on the analytical geometry of the euclidean plane and space of three dimensions. The methods of vector algebra and elementary matrix theory are used freely, but are developed from first principles, so that something like half the book is concerned with algebra. The presentation is neat and orderly. Two further volumes are promised, dealing with projective geometry and more advanced topics. J. A. T.

Sur des congruences de droites ou de courbes et sur une transformation de contact liée à ces congruences. By A. Charrueau. Pp. 72. 500 fr. 1950. Mémorial des sciences mathématiques, 115. (Gauthier-Villars)

This tract is a connected account of researches in differential geometry carried out by the author and published in several papers from 1945 to 1948. J. A. T.

The teaching of statistics in schools. A report of the Council of the Royal Statistical Society. *Journal of the Royal Statistical Society*, Series A (General), Vol. CXV, Part 1, 1952. Pp. 12. 1951. 1s.

In the *Gazette* for October, 1948 (XXXII, No. 301, p. 261) appeared a notice asking that information from members on the question of teaching statistics in schools should be sent to the Royal Statistical Society, as a committee thereof was reviewing the matter. Some teachers responded, and Mr. B. C. Brookes, who read a paper on this topic to the Association on April 10, 1947 (see *Gazette*, XXXI, No. 296, pp. 211–218) joined the Teaching of Statistics Committee. He was, I believe, the only schoolmaster on it. This Committee has already published (*J.S.S.*, CX (1), pp. 51–57, 1947) a Report on the Teaching of Statistics in Universities and University Colleges. In it they said " the statistical approach is so fundamental to the modern way of looking at things . . . that it should form part of the mental equipment of the educated man, which it is not at present. . . . We do not propose at present that universities should attempt to fill this gap in general education. . . ." In the new report they renew their plea " for the broad educative value of what has been called the statistical approach . . . we therefore urge that the subject should be introduced into all secondary schools as part of a general education ".

The Report then deals with two main problems : (i) statistics in the general curriculum ; (ii) statistics in the Sixth Form. The second aspect is largely that dealt with by Brookes in his paper, and the syllabuses reviewed are those of the various G.C.E. bodies, in which statistics is regarded as a form of applied

mathematics. The Committee comments on the extent to which the syllabuses are likely to cater for the pupil whose prime interests are not mathematical. In my opinion, such pupils, the intending biologists, medicos or economists, will find them much too mathematical. And this is my comment also on the course outlined for the " general " course. There is little appreciation of the limitations of the capabilities of the non-grammar secondary school pupils, or of those middle school grammar school pupils about whom Mr. Tuckey and Mr. Wright spoke in our discussion in 1933 (*Gazette*, XVII, No. 224, pp. 158–176). For a consideration of this problem, we may regard statistics as involving three aspects: (*a*) fundamental ideas; (*b*) computation; (*c*) mathematical analysis. The last is properly studied by the undergraduate and, in its simplest forms, by the Sixth Form specialist. The second is, in my opinion, in any serious form, impossible—you can't supply each of a class of 30 with a calculating machine at £100 apiece. We are left with the first.

The fundamental ideas are, I think, those of the " population ", of " sampling ", of probability and of correlation. The first two present great difficulties. Why have we virtually abandoned all the work that we used to do on " approximate methods "? Surely it was because the ideas of the impossibility of a measurement being absolutely correct and of repeated measurements, honestly made, still disagreeing, were too difficult for the middle form grammar school boy (or girl). The reviewer has recently, once again, tried an experimental approach with the arts people of a Sixth (grammar school) form, and with the girls of the middle school (secondary commercial), and has come again to this conclusion: the idea of a set of numbers as a population is for them a difficult idea, and they are scared if they are asked to compute any constants for it. For probability, again, for the middle school pupil, little other than some experimental work is possible; calculations and permutations and combinations frighten them. The idea behind correlation, that a functional relationship is not the only kind of relationship, is most valuable. But little more than a simple geometrical treatment of the frequency surface contours is usually feasible.

The " elementary introduction " to the statistical techniques of Section 22 of the Report is desirable for a " general education in citizenship " and could be discussed with our colleagues who teach, say, physics, geography and woodwork. But I think that it will prove that the techniques are too advanced, except in the most elementary experimental form, for any except the most capable grammar school pupils and for them only towards the end of their school life. It will be very difficult for teachers of other kinds of secondary education to do anything at all on the lines indicated. If teachers of those pupils who are so scared of figures will read this Report—and they are strongly recommended to do so—some of the statistical concepts may prove transmissible. " General education " will benefit if they do so.

FRANK SANDON.

An introduction to statistical analysis. By W. J. DIXON and F. J. MASSEY, Jr. Pp. x, 370. 38s. 6d. 1951. (McGraw-Hill)

The two authors of this book are on the mathematics staff of the University of Oregon, and their book is " essentially the same as presented in a three-quarter course for the past several years to several hundred students per year. . . . The students are from every department . . . but mainly from the physical and social sciences ".

There are nineteen chapters: " The first nine are essential to the study of the later chapters." Each chapter consists of text (sections and figures numbered from 1 onwards in each case), references, glossary, discussion ques-

tions, and class exercises and problems. The class exercises of one chapter (for example, particular samples drawn) are used again in later chapters.

The idea of Random Sampling is introduced early and there is a number of tables in the Appendix of Random Numbers and of Random Normal Numbers (for $\mu=0$, $\sigma=1$; for $\mu=2$, $\sigma=1$; for $\mu=0$, $\sigma=2$). By Chapter 6 the idea of frequency distributions has led to a discussion of bias, estimate, efficiency and confidence intervals. Chapter 7 is the first real break with the traditional text-book's early treatment. It deals with Statistical Inference, and the ideas of the Errors of Two Kinds (the error of rejecting the hypothesis when it is true, and the error of accepting the hypothesis when it is false). From this point most of the work emphasises the testing of statistical hypotheses. The hypotheses involved are those relating to means and variances, and the procedure is linked up with the use of control charts.

From Chapter 10 onwards the chapters are largely self-contained. In addition to chapters on analysis of variance and covariance, on regression and correlation, on power functions and sequential analysis, are some involving some original work of the authors and of their department. There are three in particular—those on efficiency of estimates using percentiles, the " sign " test for runs in paired observations, and the " up and down " method in " sensitivity " experiments. We might have expected some reference in this last connection to Finney's work in Probit Analysis, just as we might have expected a reference to Kendall when the authors turn to consider Rank Correlation. On the whole, however, English workers are more frequently quoted than is the case in a number of American texts. This is particularly noticeable when the very full series of tables (over 70 pages of them) is reached. The source of several of the tables (for example, of the normal curve, and of the χ^2 distribution) is not given, but E. S. Pearson's Tables for the F distribution and some of those for control charts, and his abac for confidence limits, are reproduced.

The book is well produced : the type is good and there are few misprints. The survey of the field is unusual but there are many points of value in it. The student who works through it—he will hardly be able to do so unaided—will have a sound grasp of some of the main statistical principles and he will be able to handle all the ordinary kind of practical problem (excluding such specialist problems as, for instance, the design of experiments). The price quoted above was the publication equivalent of the American cost of $4.50.

FRANK SANDON.

Mathematics for the Million. By L. HOGBEN. 3rd edition. Pp. 694. 20s. 1951. (Allen and Unwin)

In the *Gazette*, XXI, No. 243 (May, 1937), seven and a half pages were occupied by Mr. C. O. Tuckey's review of this book. In this third edition Professor Hogben continues to debunk the mystery of mathematics for the self-educator. There is a new chapter on the Algebra of the Chessboard and Card Pack (determinants, combinations and linear permutations) and the chapter on Statistics has been re-written. The Epilogue has been replaced by five appendices commenting on formulae used earlier, and answers are given to some of the examples. There are at least ten misprints in the two new chapters.

B. A. S.

Eddington's Principle in the Philosophy of Science. By SIR EDMUND WHITTAKER. Pp. v, 35. 2s. 6d. 1951. (Cambridge University Press)

Eddington was at one time a pupil of Sir Edmund Whittaker and it was Sir Edmund who edited Eddington's last book after his death in 1944. The choice of Sir Edmund to give one of the annual *Arthur Stanley Eddington Memorial*

Lectures is therefore particularly appropriate. But it is even more appropriate for the reason that, whatever the differences between the forms in which the thought of these two eminent men has been clothed, the philosophical positions reached by their thinking have very fundamental features in common.

The " principle " in the title is, in Whittaker's words : *All the quantitative propositions of physics, that is, the exact values of the pure numbers that are constants of science, may be deduced by logical reasoning from qualitative assertions, without making any use of quantitative data derived from observation.*

Whittaker first sketches the historical evolution of the principle which he shows, in particular, to accord with the philosophical outlook initiated by Leibnitz. He also touches upon the mathematical basis of Eddington's work, making special reference to the E-tensor calculus recently introduced in this connexion by S. R. Milner. but apparently not yet published. He then passes, via a general assessment of the status of theoretical physics, to consider certain possible objections to Eddington's principle.

Perhaps the most serious of these objections is that valid quantitative predictions can actually be made by using qualitative conceptions that are known to be impermanent. Whittaker has some interesting considerations to offer, by way of overcoming this objection, concerning the possibility of reducing all the required qualitative assertions to *postulates of impotence*, which may be deemed to be of more permanent acceptability. (*Cf.* E. T. Whittaker, *Proc. Roy. Soc. Edin.*, **61** (1941), 160.)

Whittaker then gives a careful statement of Eddington's attitude towards the " constants of nature ". His consideration of Eddington's " cosmical number " leads him to some very penetrating comments on the relation of Eddington's work to recent cosmological theories involving the continuous creation of matter.

Finally, Whittaker broadens his survey to consider whether " it throws any light upon recent presentations of the philosophy of religion ". Since he thereupon refers to the work of A. N. Whitehead, he finds a natural passage to this topic. The central consideration to which he calls attention in this context is, " The fact that changes in our material universe can be predicted—that they are subject to mathematical law—is the most significant thing about it, for mathematical law is a concept of the mind, and from the existence of mathematical law we infer that our minds have access to something akin to themselves that is in or behind the universe."

This brief sketch will serve to show that the lecture covers an immense range of ideas. It is rich in illuminating remarks concerning a great number of the most profound problems of philosophy and epistemology. Any possibly useful comments on the ideas propounded would have to be far too lengthy to be given here. It is much better to recommend reading and pondering upon the lecture itself. It need scarcely be added that it is presented with the incisive brilliance of everything written by Sir Edmund Whittaker.

W. H. McCrea.

The Theory of the Riemann Zeta-functions. By E. C. Titchmarsh. Pp. 346. 40s. 1951. (Oxford University Press)

The main incentive to a study of the Riemann ζ-function is the still undecided Riemann hypothesis dating from 1859 that

$$\zeta(s) \neq 0 \quad \text{for} \quad R(s) > \tfrac{1}{2}.$$

When the author published his first book on the subject in 1930 important advances towards a proof of the hypothesis had been made by Hadamard, De La Vallée-Poussin, Bohr-Landau, Hardy, Littlewood and many others. Since then further important work has been done, mainly by Vinogradov and

his school and lately by A. Selberg. But the problem has revealed defences in depth probably not anticipated by earlier investigators, and the law of diminishing returns seems to bog down the most ingenious attacks of the later assailants.

Compared with the original treatise of the author, a sometimes rather sketchy Cambridge Tract, the book is now a solid self-contained volume which represents the latest state of science. The most important additions to the first edition are :

1. Vinogradov's results on exponential sums and the application to the order of magnitude of $\zeta(s)$ and the error term in the prime number theorem.
2. A. Selberg's theorem that the number of zeros on the critical line has at least order of magnitude $\geqslant$ const. $T \log T$.
3. A chapter on divisor problems.
4. A report on numerical calculations of the zeros on the critical line.

An excellent bibliography lists all papers on the subject since 1909, when Landau's *Handbuch der Lehre von der Verteilung der Primzahlen* was published.

The reviewer would like to direct the expert's attention to theorem 9.14 on p. 194 which has not been published hitherto, and expresses his regret that no reference has been made to the work of Wiener on Tauberian theorems which appears to him to be the least unhopeful way of having another shot at a proof of the Riemann hypothesis. H. HEILBRONN.

Kurze Mathematiker-Biographien. Beihefte zur Zeitschrift *Elemente der Mathematik*, Nos. 2–11. 24 pp. each. Sw. fr. 3.50 each. (Birkhäuser, Basel)

Many teachers will wish to supplement the brief biographies in the standard histories of mathematics by more detailed accounts of their own favourites. The present series of biographical supplements to the recently founded Swiss journal, *Elemente der Mathematik*, provides an admirable assistance in this direction. The life and work of each mathematician is described, in a beautifully printed brochure, with portraits and facsimiles. We hope that many more names will be added to the present ten, and that teachers will add these tracts to their private or school shelves.

In the following list of titles and authors, the language, French or German, of the text is indicated by F or G.

Jakob Steiner (L. Kollros, F) ; Leonhard Euler (R. Fueter, G) ; Ludwig Schläfli (J. J. Burckhardt, G) ; Jost Bürgi und die Logarithmen (E. Voellmy, G) ; Johann und Jakob Bernoulli (J. O. Flekenstein, G) ; Evariste Galois (L. Kollros, F) ; Niels Henrik Abel (O. Ore, F) ; Gaspard Monge (R. Taton, F) ; Pierre Fermat (J. Itard, F) ; Die beiden Bolyai (L. v. Dávid, G).

T. A. A. B.

Anschauliche Geometrie. By D. HILBERT and S. COHN-VOSSEN. Pp. x, 314. \$3.95.

Die mathematischen Hilfsmittel des Physikers. By E. MADELUNG. 3rd edition. Pp. xiii, 384. \$4.50 (Dover Publications, New York)

As books increase in price, the knowledge of what classic texts are available in reprints at moderate prices becomes increasingly important. The courtesy of Dover Publications allows us to draw attention to two such reprints, each of them an essential item in any scientific library.

The brilliant intuitive and concrete approach to geometry provided by Hilbert and his collaborator Cohn-Vossen needs little recommendation. Six chapters (Simple curves and surfaces, regular systems of points, configurations, differential geometry, kinematics, topology) serve to lead the average reader to a number of vantage points from which large domains can be surveyed. The authors have Clifford's gift of carrying the reader in a few short strides

from the elements to the very heart of a geometrical topic, and their knack of exhibiting an underlying unity connecting and elucidating apparently dissimilar regions, a knack which the novice cannot too early begin to cultivate, is in the best Klein tradition. 330 illustrations keep the exposition simple and alive ; verisimilitude in three-dimensional diagrams is heightened by careful use of thick, thin and tapering lines, and the reprint reproductions are excellent, save perhaps for a slight deterioration in the photographs of models. No better stimulus for the young geometer could be found.

Madelung's compendium of pure and applied mathematical formulae is well-established as a work of reference. The new (4th) German edition is now on sale, but additions to the contents have entailed an increase in price. This reprint of the 3rd German edition is therefore a convenient and economical "utility model". 12 sections summarise pure mathematics, and 6 sections give the bones of dynamics, electrodynamics, relativity, quantum theory, thermodynamics and statistical theories. A non-specialist who must suddenly recall the Maxwell-Boltzmann equation, or the recurrence relations for the Laguerre polynomials, will find them in Madelung, with some indication of where to go for more detailed information.

A useful feature of each volume is a German-English Glossary.

T. A. A. B.

Ninth Astronomer Royal. The Life of Frank Watson Dyson. By MARGARET WILSON. Pp. xvi, 294, with frontispiece and 13 plates. 25s. 1951. (Cambridge : Heffer)

The list of Astronomers Royal is not a long one, considering that the first was appointed by Charles II in 1675. On putting down this book one is left wishing that we had similar records of Dyson's predecessors. Would the towering figure of Airy, for example, seem any less grim and forbidding if we had some more intimate record to remember him by than the ponderous prose of his scientific papers and—dare one say it—the Victorian pomposity of his autobiography? Future generations will be better served in respect of Dyson : Mrs. Wilson has here set down with loving care a picture of the man which those who knew him will acclaim as authentic.

The name of Dyson is one to conjure with still in Greenwich, both in the town to whose corporate life he unsparingly devoted his scanty spare time and in the Observatory where he spent thirty-four years of his life, first as Chief Assistant and then as Director. The reasons emerge clearly enough from this charming biography written by the fourth of his six daughters : they are preeminently his quiet unassuming friendliness and hospitality and his unbounded enthusiasm for any cause he took up. His whole life was of course bound up with astronomy, yet his professional activities figure here only as background to the story of his life with his gracious wife and large family. This is primarily a sympathetic record, told by one in its inner circle, of the evolution of a closely-knit middle-class family headed by a man and woman whose good fortune it was to inspire respect and affection—yes, and love too—in those they lived among. Though she claims no technical knowledge, the author has done no violence to her father's astronomical work, and her book should make its mark as a sensitive record amply documenting a period of rapid social change.

Packed as it is with good anecdotes of mathematicians, astronomers and other scientists of the turn of the century (who can resist the story of Dyson's part in the *Granta's* Mathematical Tripos "scoop" in 1892?), this book will appeal not merely to those who knew Dyson—they will read it anyway—but also to a much wider circle of readers who can be trusted to appreciate an unassuming biography that is also a piece of good writing. A. H.

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